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Risk preference heterogeneity in group contests

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ABSTRACT

We analyze the first model of a group contest with players that are heterogeneous in their risk preferences. In our model, individuals' preferences are represented by a utility function exhibiting a generalized form of constant absolute risk aversion, allowing us to consider any combination of risk-averse, risk-neutral, and risk-loving players. We begin by proving equilibrium existence and uniqueness under both linear and convex investment costs. Then, we explore how the sorting of a compatible set of players by their risk attitudes into competing groups affects aggregate investment. With linear costs, a balanced sorting (i.e., minimizing the variance in risk attitudes across groups) always produces an aggregate investment level that is at least as high as an unbalanced sorting (i.e., maximizing the variance in risk attitudes across groups). Under convex costs, however, identifying which sorting is optimal is more nuanced and depends on preference and cost parameters.

1. Introduction

In a *group contest*, such as research teams competing for a grant, workplace competitions, political campaigns, R&D tournaments, lobbying, and team sporting events, individuals belonging to a group make irreversible investments (e.g., effort, time, or resources) with the hopes of securing a prize (e.g., monetary reward or economic rent) for their group.[1] One important feature of many contests is that the winner determination process is inherently noisy, implying that the winning group may or may not be the group with the highest investment. Instead, winning is often better expressed *probabilistically*, with the intuitive properties that the likelihood of winning is increasing in own group investment level, and decreasing in the investment levels of other groups.[2] Given that competing in a group contest is in essence a *gamble*, it is immediately of interest to explore how *risk preference heterogeneity* impacts contestants' behavior.

In this paper, we present and analyze the first model of a group contest in which groups are comprised of individuals with heterogeneous attitudes towards risk. In our setting, individuals' preferences are represented by a utility function exhibiting a generalized form of constant absolute risk aversion (CARA), allowing us to explore arbitrary configurations of risk-averse, risk-neutral, and risk-loving individuals within and between groups. Apart from the significant complication of risk preference heterogeneity, the group contest setting we consider is simple. Individual group members simultaneously and independently make costly investments, within-group investment aggregation follows a perfectly substitutable production technology, a single group-specific public-good prize is contested, and a group's probability of winning the prize is determined by Tullock's imperfectly discriminating contest success function with parameter $r = 1$ (Tullock, 1980).

Our paper is organized into two parts. In the first part, we establish equilibrium existence in the group contest model discussed above under both linear and convex investment costs. To the best of our knowledge, we present the first systematic study and equilibrium analysis in *any* group contest setting beyond the risk-neutral player benchmark. Given a linear cost function, we derive in closed-form players' participation conditions and obtain group share functions from the representative players' share functions in each group. While aggregate investment in equilibrium is unique, each individual investment is not unique provided there are multiple representative members within a group. In such a case, not only do the non-representative members free ride, but also some of the representative members.

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1 For pioneering contributions to the theoretical literature on group contests see, e.g., Katz et al. (1990), Nitzan (1991), Baik (1993), Nti (1998), Esteban and Ray (2001), and Baik (2008). See Sheremeta (2018) for a review of the empirical group contest literature.

2 Such noise may be present, for example, in organizational settings with imperfect managerial monitoring and/or when it is difficult for the manager to quantitatively and precisely measure work efforts, or noise may simply reflect a random "luck" component. For example, output, y, can be written as an additively separable function effort, e: $y = e + \eta$, where η is an *i.i.d.* random shock or luck parameter (Lazear and Rosen, 1981). Depending on the variance of noise parameter η, output, and hence winning probabilities, can be more or less sensitive to effort.

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However, with convex investment costs, even though a closed-form solution is not obtainable, we show that there exists a unique equilibrium in which all players participate in the contest with nonzero investments.

Before exploring the consequences of preference heterogeneity on contest behavior, we first characterize equilibrium behavior in two important cases involving player symmetry that yield a symmetric equilibrium at the group level. In the first case, all players' risk preferences are identical. In equilibrium, all groups are active, all players are representative, and group investments are identical. Interestingly, we find that there exists a unique symmetric risk preference in the nonnegative domain of risk preference parameters (i.e., when players are either risk neutral or risk loving) that maximizes aggregate investment. In the second case, the least risk-averse player is identical across groups, but all other players' risk preferences are arbitrary. In this case, when the least risk-averse group member is not risk loving, all groups are active and only the least risk-averse players are representative. This equilibrium is preserved even when the least risk-averse members of each group are risk loving, but the number of groups must be sufficiently high.

In the second part of the paper, we investigate how player heterogeneity, both within and between groups, affects investment levels. We not only characterize how individuals behave based on own risk attitude and others' risk attitudes, but also how a group's "risk preference" impacts group and aggregate investment. While it is natural to investigate heterogeneity in this way, comparative static predictions, such as establishing the relationship between aggregate investment and the overall degree of contestant risk heterogeneity, are often difficult or impossible to obtain in such a generalized setting. We approach this challenge by focusing most of our analysis in this part on one interesting aspect of heterogeneity in group contests. Specifically, we explore the optimal *sorting* of heterogeneous players, and ask the following questions: For a given set of players with heterogeneous risk preferences, how should the players be sorted—*exogenously assigned*—into competing groups so as to maximize aggregate effort? For example, if a manager wishes to hold a competition between groups of salespeople with the goal of maximizing total sales, how should she sort various risk attitudes into competing groups?[3] Should she create a *balanced* sorting, whereby workers are "evenly" distributed across groups such that overall group risk preferences are similar across groups, or should she instead create an *unbalanced* sorting, in which similar risk attitudes are pooled together so that one group contains the least risk-averse workers, another group contains the most risk-averse workers, and everything in between? To make each sorting comparable, we assume a compatible set of players such that when the manager uses a balanced sorting she creates identical groups consisting of heterogeneous workers, and when she uses an unbalanced sorting she creates heterogeneous groups consisting of identical workers.

Sorting is especially relevant presently, as organizations have increasingly adopted team-based relative performance incentives over the last few decades.[4] Furthermore, recent findings from laboratory experiments support the adoption of team-based incentives, with some studies finding that team contests stimulate higher efforts than contests between individuals (Chen and Lim, 2013; Li et al., 2019). So far, however, the theoretical literature on optimal sorting in contests is limited to two studies, each considering the case of risk-neutral players with ability heterogeneity.[5] From a contest design perspective, sorting personnel into competing groups is a low-cost or costless operation for a manager, but as we show, the impact of sorting on aggregate effort is nontrivial. In practice, it may be difficult to obtain accurate cardinal measures of individual characteristics, such as ability or risk attitude. Sorting, however, only relies on ordinal rankings, which are generally easier to obtain. Ordinal rankings can be acquired directly from historical data on past performance and through repeated interactions, or inferred indirectly via data that are correlated with the characteristic being ranked. Specifically, the degree of one's risk aversion has been found to be significantly correlated with easy-to-observe individual characteristics such as age (see, e.g., Albert and Duffy, 2012) and sex (see, e.g., Borghans et al., 2009; Croson and Gneezy, 2009; Charness and Gneezy, 2012). Moreover, many organizations are already routinely collecting risk attitude metrics, among others, by means of psychometric testing completed during job applicant screening; risk-based sorting is particularly well-suited in such organizations.

When costs are linear, we show that a balanced sorting always produces an aggregate investment level at least as high as an unbalanced sorting. However, when costs are convex, our exploratory numerical simulations show that the optimal sorting may be balanced or unbalanced, and the dominance of one sorting over the other depends on the average measure of risk aversion across players and its variation, and also the cost function. To explore this further, we restrict our attention to the case of *weak heterogeneity* in risk preference parameters, and employ a regular perturbation problem approach in order to find an approximate, but explicit representation of aggregate investment. By weak heterogeneity, we refer the case when players' risk attitudes are closely centered about a given average measure of risk aversion. Such weak forms of heterogeneity are arguably the most interesting cases to explore, because naturally occurring forces, such as occupation self-selection, tend to pool together individuals with similar attributes. In the quadratic approximation, we show that the optimal sorting of players critically depends on the sample variance in risk attitude across players and across groups and the cost function. Under quadratic costs, the balanced sorting is optimal when all players are risk averse, but it is possible that the unbalanced sorting is optimal when players are risk loving. The optimal sorting is even more nuanced under a general class of convex cost functions.

Our paper is organized as follows. In Section 2, we summarize the literature exploring the impact of various risk preferences on behavior in contests. In Section 3, we present our model of a group contest with risk preference heterogeneity and analyze equilibrium existence and uniqueness. In Section 4, we explore the effects of heterogeneity on investment levels and discuss the optimal sorting of players. We discuss our results and conclude in Section 5.

[3] Team sales competitions are one of the most commonly utilized forms of workplace incentives. Large companies such as Dunkin' Donuts (O'Keeffe et al., 1984) and E-Mart Everyday (https://www.agweb.com/article/sales_competition_boosts_us_beef_at_korean_grocery_chain_NAA_News_Release) have used sales competitions with great success.

[4] See Lawler et al. (1995, 2001), Lazear and Shaw (2007).

[5] Ryvkin (2011) finds that a balanced sorting produces higher investments compared to an unbalanced sorting in a Tullock group contest with perfectly substitutable within-group investments, but this can be reversed for sufficiently "steep" convex investments costs. Brookins et al. (2015b) generalize Ryvkin (2011) by considering a generalized CES production technology for the overall investment level of a group. They find that the optimal sorting now depends not only on the steepness of convex investment costs, but also on the degree of investment substitutability within each group. Brookins et al. (2015a, 2018) test many of these predictions using a laboratory experiment. They find that, regardless of the degree of substitutability within each group, a balanced sorting of players generates significantly higher investments as compared to an unbalanced sorting.

2. Beyond risk neutrality in contests

While the contest literature is large, the subset of papers examining behavior beyond the risk-neutral paradigm is small and dedicated to individual contests.[6] In particular, theoretical research of risk aversion in contests is rare because an equilibrium may not uniquely exist given nonlinear utility functions (Skaperdas and Gan, 1995), and most comparative statics results are ambiguous (see, e.g., Konrad and Schlesinger, 1997). Moreover, several recent papers point out that behavior not only depends on risk aversion, but also on prudence. Even though prudence has been defined as an incentive to save in the precautionary saving problem (Kimball, 1990), it is equivalent to aversion to downside risk.[7] Treich (2010) and Jindapon and Whaley (2015) study the effects of prudence in individual contests given risk-averse and risk-loving players, respectively. In symmetric contests, Treich (2010) shows that risk-aversion decreases rent-seeking efforts under prudence while Jindapon and Whaley (2015) find that risk lovingness and imprudence jointly increase investments. Sahm (2017) and Jindapon and Yang (2017) theoretically study individual contests with heterogeneous risk preferences, while March and Sahm (2018) experimentally explore the effects of risk aversion and prudence in contests with ability heterogeneity.

In this paper, the existence and uniqueness of equilibrium is established using a generalized "share function approach", which was developed and utilized by Cornes and Hartley (2003) and Cornes and Hartley (2012) to prove existence and uniqueness of equilibrium in contests between individuals with risk-averse players.[8] More recently, this approach has proved useful in existence and uniqueness analyses in contests between individuals with heterogeneous risk-loving players (Jindapon and Whaley, 2015) and heterogeneous players exhibiting generalized CARA preferences (Jindapon and Yang, 2017), as well as in a group contest setting with risk-neutral players with heterogeneous abilities (Nitzan and Ueda, 2014) and different prize allocations within the winning group (Trevisan, 2020).[9] For the linear investment cost case, we derive each player's desired probability of winning for her group, and refer to the players with the highest desired probability of winning in each group as representative members. Thus, each group's share function is derived from the share functions of the group's representative members and the equilibrium is found where the sum of all group shares is equal to one. Like the equilibrium structure in a group contest under risk neutrality, players who are not representative members will not contribute to their group because it is their best response. However, we find that representative members are not necessarily the least risk-averse players in the group—contrasting the literature of group contests where players with the highest ability or prize valuation are the representative members. In case of convex investment cost, each group's share function is the solution of a system of individual share functions within the group given that the sum of all individual shares is equal to one. We then derive an equilibrium by finding the aggregate investment level such that the sum of all group shares is equal to one.

One benefit of focusing on CARA preferences is that it is more tractable than adopting other concave or convex utility functions. Since each CARA player's initial wealth does not affect his best response, like all other contest models in the literature assuming risk neutrality, this framework allows us to derive an explicit solution and perform comparative statics with respect to the players' risk attitudes.[10] Moreover, we believe that the generalized CARA form is consistent with many empirical studies in the literature because all players in our model are prudent (see Footnote 7). In laboratory experiments, Deck and Schlesinger (2014), Ebert and Wiesen (2014), Noussair et al. (2014) all find that most subjects are prudent regardless of whether they are risk averters or risk lovers.

We believe that our theoretical framework is essential to continue to build on our understanding of the consequences of various risk preferences in contest settings, and extend this knowledge beyond the individual contest case. In a review of the experimental literature on group contests, Sheremeta (2018) points out that Abbink et al. (2010) (see Appendix B) is the only study – considering both the experimental and theoretical literature – to explore the consequences of risk aversion on investment. Assuming players are either symmetrically risk-averse or risk-neutral, the authors show that investment levels are unambiguously decreasing in risk aversion under CARA and constant relative risk aversion (CRRA) preferences. However, existence and uniqueness results are absent. This paper serves as the first equilibrium analysis of *any* group contest model with nonlinear utility functions, and we push far beyond the aforementioned symmetric player benchmark. Consistent with Abbink et al. (2010), when players are symmetric, we show that investment levels are unambiguously decreasing in risk aversion relative to the risk neutral benchmark. However, this finding does not generally hold when risk loving preferences are possible. Indeed, we show that aggregate investment is single-peaked in the symmetric risk level, and prove that the unique maximizer is strictly increasing in the number of groups.

3. The model

Consider a group contest between $n \geq 2$ groups, each consisting of $m \geq 1$ players. Players have an initial wealth $I > 0$. Each player ik simultaneously and independently chooses an investment level, $x_{ik} \geq 0$, to increase her group's chance of winning the contest. A prize of value $V > 0$ is awarded to each member of the winning group. Assuming an imperfectly discriminating contest success function (Tullock, 1980), the probability that group i wins the contest is given by

$$p_i = \frac{X_i}{X}, \quad (1)$$

[6] We focus our attention in this section on studies utilizing the Tullock-style (Tullock, 1980) imperfectly discriminating contest success function. For general reviews of the theoretical contest literature see, e.g., Congleton et al. (2008), Konrad (2009), Connelly et al. (2014), and Corchón and Serena (2018); for a review of the experimental contest literature see Dechenaux et al. (2015). The interplay between risk aversion and contest expenditure has been discussed in other contest settings. For rank-order tournaments (Lazear and Rosen, 1981) see Green and Stokey (1983), Nalebuff and Stiglitz (1983), Krishna and Morgan (1998), Kalra and Shi (2001), Akerlof and Holden (2012). For all-pay auctions with complete information (Baye et al., 1996) see Hillman and Samet (1987), Siegel (2009), Chen et al. (2017). Finally, for all-pay auctions with incomplete information (Moldovanu and Sela, 2001) see Fibich et al. (2006), Parreiras and Rubinchik (2010).

[7] Menezes et al. (1980) define an increase in downside risk as a third-order stochastic deterioration while the mean and variance are preserved. Under expected utility, downside risk aversion (or prudence) is equivalent to positive third derivative of a utility function. Even though the generalized CARA functional form allows the players to be risk averse or risk loving, all players are downside risk averse. Eeckhoudt and Schlesinger (2006) define prudence in the risk apportionment framework without the assumption of expected utility.

[8] See Cornes and Hartley (2005) for an application of the share function approach to contests between individuals with ability heterogeneity.

[9] While the share function approach can accommodate heterogeneity in the cost of investment (ability) and prize valuation, we omit this further complication to focus solely on preference heterogeneity.

[10] Given a nonlinear utility function, some comparative statics with respect to each player's initial wealth can be obtained by assuming nonmonetary rent or effort as shown in Schroyen and Treich (2016).

where $X_i = \sum_{k=1}^{m} x_{ik}$ and $X = \sum_{i=1}^{n} X_i$.[11] For completeness, assume $p_i = \frac{1}{n}$ if $x_{ik} = 0$ for all i, k. Investments are costly for each player and characterized by a strictly increasing cost function $g(x_{ik})$. Let $u_{ik}(W_{ik})$ be player ik's utility given wealth W_{ik}. Therefore, player ik's expected utility given investment level $x_{ik} \geq 0$ is

$$Eu_{ik} = p_i u_{ik}(I + V - g(x_{ik})) + (1 - p_i)u_{ik}(I - g(x_{ik})). \tag{2}$$

Note, that any player can guarantee herself a payoff of $Eu_{ik} = u_{ik}(I)$ with an investment equal to zero. Thus, a necessary condition for participation in the contest requires that the following inequality is satisfied[12]

$$p_i u_{ik}(I + V - g(x_{ik})) + (1 - p_i)u_{ik}(I - g(x_{ik})) \geq u_{ik}(I). \tag{3}$$

Assume players' preferences over wealth W_{ik} are described by constant absolute risk aversion (CARA), and take the form

$$u_{ik}(W_{ik}) = \begin{cases} \frac{e^{\alpha_{ik} W_{ik}} - 1}{\alpha_{ik}} & \text{if } \alpha_{ik} \neq 0 \\ W_{ik} & \text{if } \alpha_{ik} = 0 \end{cases} \tag{4}$$

We allow for heterogeneity in the players' degree of risk aversion (or risk lovingness), implying α_{ik} and α_{jl} can be different for any two players ik and jl.

3.1. Linear cost

First, we assume that $g(x)$ is linear.

Assumption 1. $g(x) = cx$ where $c > 0$.

If player ik is risk neutral, i.e., $\alpha_{ik} = 0$, the first-order condition for player ik is given by

$$\frac{X - X_i}{X^2} V - c = 0. \tag{5}$$

Given $X_{-i} := X - X_i$ and $x_{-ik} := X_i - x_{ik}$, we can derive player ik's optimal investment as her best response to X_{-i} and x_{-ik} from (5). We find that there exist threshold values of X_{-i} and x_{-ik}, denoted by T_{ik} and t_{ik} respectively, such that player ik optimally chooses $x_{ik} = 0$ whenever $X_{-i} \geq T_{ik}$ or $x_{-ik} \geq t_{ik}$. If the aggregate investment of all other groups or X_{-i} is larger than T_{ik}, then group i's best response (from player ik's personal view) is to leave the contest. This is analogous to individual contests, but in group contests, group i may not actually leave the contest because some other group members still contribute. If the aggregate investment of all other members of group i, or x_{-ik}, is larger than t_{ik}, then player ik's best response is to choose $x_{ik} = 0$ and free ride. We call T_{ik} and t_{ik} player ik's between-group and within-group participation threshold respectively. According to (5), we find that

$$T_{ik} = \frac{V}{c} \tag{6}$$

and

$$t_{ik} = -X_{-i} + \sqrt{T_{ik} X_{-i}} \tag{7}$$

whenever player ik is risk neutral, i.e., $\alpha_{ik} = 0$.

If player ik is risk averse or risk loving, i.e., $\alpha_{ik} \neq 0$, the first-order condition for player ik is given by

$$\frac{X - X_i}{X^2} \left(\frac{e^{\alpha_{ik} W_{ik}^V} - 1}{\alpha_{ik}} - \frac{e^{\alpha_{ik} W_{ik}^L} - 1}{\alpha_{ik}} \right) - c \left(\frac{X_i}{X} e^{\alpha_{ik} W_{ik}^V} + \frac{X - X_i}{X} e^{\alpha_{ik} W_{ik}^L} \right) = 0 \tag{8}$$

where $W_{ik}^V = I + V - cx_{ik}$ and $W_{ik}^L = I - cx_{ik}$. It follows that player ik will not participate in the contest, i.e., choosing $x_{ik} = 0$, if $X_{-i} \geq T_{ik}$, where

$$T_{ik} = \frac{e^{\alpha_{ik} V} - 1}{\alpha_{ik} c}, \tag{9}$$

or $x_{-ik} \geq t_{ik}$, where

$$t_{ik} = \frac{-(e^{\alpha_{ik} V} + 1)X_{-i} + \sqrt{(e^{\alpha_{ik} V} - 1)^2 X_{-i}^2 + 4e^{\alpha_{ik} V} T_{ik} X_{-i}}}{2e^{\alpha_{ik} V}}. \tag{10}$$

Note that if we write T_{ik} and t_{ik} in (9) and (10) as a function of α, we find that $\lim_{\alpha \to 0} T_{ik} = T_{ik}$ in (6) and $\lim_{\alpha \to 0} t_{ik} = t_{ik}$ in (7). In addition, $t_{ik} > 0$ if and only if $X_{-i} < T_{ik}$. Therefore, player ik's best response to X_{-i} and x_{-ik} is to be inactive whenever (i) $X_{-i} \geq T_{ik}$ or (ii) $X_{-i} < T_{ik}$ and $x_{-ik} \geq t_{ik}$. Given $X_{-i} < T_{ik}$, only players with the highest value of t_{ik} have an incentive to participate in the contest and they will choose x_{ik} so that $X_i = t_{ik}$.

Given T_{ik} and t_{ik} in (9) and (10), we find that T_{ik} is increasing in α_{ik} while the effect of α_{ik} on t_{ik} is ambiguous. For example, suppose that $V = c = 1$. We can plot T_{ik} and t_{ik} on α_{ik} as in Fig. 1. If there are three players in group i with $\alpha_{i1} = -2$, $\alpha_{i2} = 0$, and $\alpha_{i3} = 2$, then $T_{i1} < T_{i2} < T_{i3}$. Given $X_{-i} = 0.25$, we find that $X_{-i} < T_{ik}$ for $k = 1, 2, 3$ (see Fig. 1(a)) and that $t_{i2} > t_{i3} > t_{i1}$ (see Fig. 1(b)). Player $i2$ will choose $x_{i2} = t_{i2}$ and become the only active player of group i because investing zero is the other two players' best response to $x_{i2} = t_{i2}$. However, when $X_{-i} = 1$, both players $i1$ and $i2$ will not participate because $X_{-i} \geq T_{i1}, T_{i2}$. Player $i3$ will choose $x_{i3} = t_{i3}$ as the best response to $X_{-i} = 1$ and $x_{i1} = x_{i2} = 0$.

Following Cornes and Hartley's (2003, 2005, 2012) share function approach, we define individual share function $s_{ik}(X) := \frac{X_i}{X}$ as player ik's desired probability of winning the contest for her group according to the first-order condition. Given $\alpha_{ik} = 0$, (5) implies

$$s_{ik}(X) = \begin{cases} 1 - \frac{X}{T_{ik}} & \text{if } X < T_{ik} \\ 0 & \text{if } X \geq T_{ik} \end{cases} \tag{11}$$

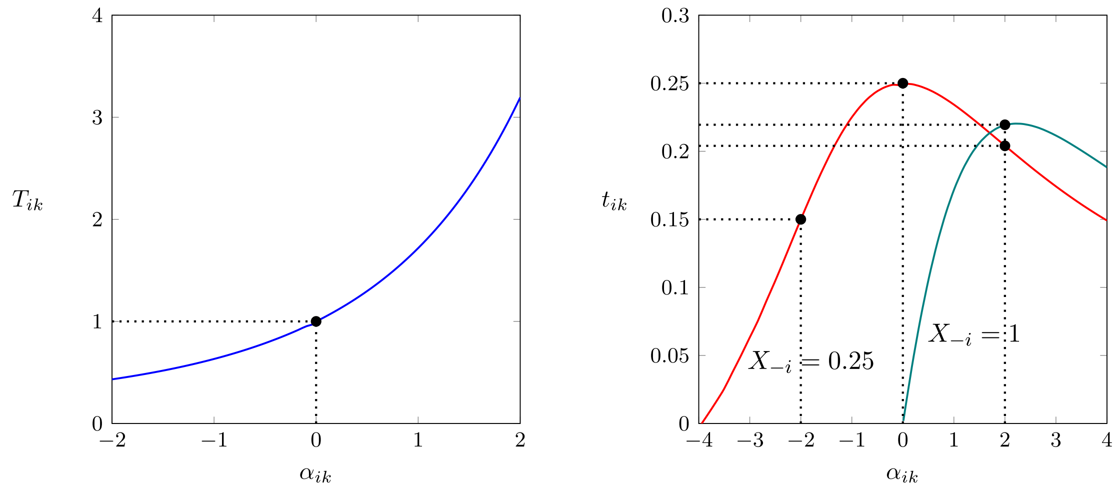
where T_{ik} is given by (6). Given $\alpha_{ik} \neq 0$, (8) implies

$$s_{ik}(X) = \begin{cases} \frac{T_{ik} - X}{T_{ik} + (e^{\alpha_{ik} V} - 1)X} & \text{if } X < T_{ik} \\ 0 & \text{if } X \geq T_{ik} \end{cases} \tag{12}$$

where T_{ik} is given by (9). Next, we define group i's share function $s_i(X)$ as group i's probability of winning given mutual best responses of all the players within the group. Suppose that $m = 2$ and $s_{i1}(X) > s_{i2}(X)$ for some value of X. Player $i1$ has a stronger desire to increase group i's probability of winning than does player $i2$. If player $i2$ chooses x_{i2} so that group i's probability of winning is $s_{i2}(X)$, player $i1$ will add a positive investment so that group i's probability of winning rises to $s_{i1}(X)$. Knowing that player $i1$ will make the group's investment higher than the desired group level from the perspective of player $i2$, player $i2$'s best response to player $i1$'s strategy is to invest zero and free ride. As a result, player $i1$ is the only contributor so that $s_i(X) = s_{i1}(X)$.

[11] We assume that within-group investment is perfectly substitutable, but other technologies have been considered. In Tullock group contests, Kolmar and Rommeswinkel (2013) and Brookins et al. (2015b) assume a generalized CES production function; Lee (2012) and Baik and Shogren (1998), Chowdhury et al. (2013) explore weak-link and best-shot production functions, respectively; Chowdhury and Topolyan (2016) consider a group contest in which one group has a best-shot technology and the other weak-link; and these aggregation functions have also been used in the group all-pay auction literature (see, e.g., Baik et al., 2001; Topolyan, 2014; Chowdhury et al., 2016; Barbieri et al., 2014); however, all aforementioned studies only consider the case of risk-neutral players.

[12] This condition is necessary but not sufficient. In the linear cost case, we derive sufficient conditions for player ik from $X_{-i} := X - X_i$ and $x_{-ik} := X_i - x_{ik}$. In the convex cost case, this condition holds for all ik in equilibrium.



(a) Between-group threshold (b) Within-group threshold

Fig. 1. Between-group and within-group participation thresholds given a linear cost function.

For any $m \geq 2$, we use the same reasoning to derive group i's share function given X as

$$s_i(X) = \max\{s_{i1}(X), \dots, s_{im}(X)\}. \tag{13}$$

If $s_{ik}(X) \geq s_{il}(X)$ for all $l \neq k$, we say that player ik is a representative member of group i given X and define $R_i(X)$ as the set of all representative members of group i given X. It follows that group i's aggregate investment, X_i, corresponding to $s_i(X)$ is the sum of x_{ik} for all $ik \in R_i(X)$ and players who are not in $R_i(X)$ free ride. Note that if there are multiple representative members, it is possible that some representative members free ride as well. We can derive important properties of $s_i(X)$ as in Lemma 1.

Lemma 1. *Let $T_i := \max\{T_{i1}, \dots, T_{im}\}$. Under Assumption 1, group i's share function $s_i(X)$ is a continuous function given $X > 0$ with the following properties.*

(i) $s_i(X) = 0$ for all $X \geq T_i$.
(ii) $s_i(X)$ is strictly decreasing in X for all $X \in (0, T_i)$.
(iii) $\lim_{X\to 0} s_i(X) = 1$ and $\lim_{X\to T_i} s_i(X) = 0$.

Proof. These properties follow from (11), (12), and (13). ■

Now we define

$$s(X) := \sum_{i=1}^{n} s_i(X) \tag{14}$$

as the sum of all group shares. Let $T := \max\{T_1, \dots, T_n\}$. It follows immediately from Lemma 1 that $s(X)$ is a continuous function given $X > 0$ with the following properties.

(i) $s(X) = 0$ for all $X \geq T$.
(ii) $s(X)$ is strictly decreasing in X for all $X \in (0, T)$.
(iii) $\lim_{X\to 0} s(X) = n$ and $\lim_{X\to T} s_i(X) = 0$.

The above properties guarantee that there exists a unique value of $X \in (0, T)$ such that $s(X) = 1$. In other words, this is the only value of aggregate investment by all players in the contest that makes the sum of all groups' probabilities of winning equal to one. Thus, it is the aggregate investment in equilibrium and we call it X^e. We let X_i^e and x_{ik}^e represent group i's aggregate investment and player ik's investment in equilibrium, respectively, and define vectors $\mathbf{X}^e := [X_1^e, X_2^e, \dots, X_n^e]$ and $\mathbf{x}_i^e := [x_{i1}^e, x_{i2}^e, \dots, x_{im}^e]$. It follows that group i's investment in equilibrium is $X_i^e = s_i(X^e)X^e$. Since X^e is unique, the corresponding vector $\mathbf{X}^e$ is also unique. If $s_i(X^e) = 0$ for some i, then $X_i = 0$ and no one in group i makes an investment in equilibrium. Given $s_i(X)$ in (13), we know that any vector $\mathbf{x}_i^e$ such that $\sum_{k=1}^m x_{ik} = X_i^e$ and $x_{ik} = 0$ for all $ik \notin R_i(X^e)$ constitutes an equilibrium. Thus, all non-representative members of group i are inactive in equilibrium. If $s_i(X^e) > 0$ and there are multiple representative members of group i given X^e, it is possible that some of them are also inactive in equilibrium. Thus, all of the non-representative members and the inactive representative members completely free ride on the active representative members of group i. If group i has only one representative member given X^e, then that member is the only contributor of the group. It follows that vector $\mathbf{x}_i^e$ has only one non-zero entry and it is unique.

Proposition 1. *Under Assumption 1, there exists a pure strategy Nash equilibrium. The aggregate investment in equilibrium, X^e, is unique. The corresponding vector $\mathbf{X}^e$ is unique. If $R_i(X^e)$ is singleton, then the corresponding vector $\mathbf{x}_i^e$ is also unique.*

Proof. See the argument above. ■

We obtain the following properties in equilibrium.

Corollary 1. *Consider an equilibrium in which X^e is the aggregate investment under Assumption 1.*

(i) There are at least two active groups.
(ii) If i is an active group, then there exists player ik such that $T_{ik} > X^e$. If i is an inactive group, then $T_{ik} \leq X^e$ for all $k = 1, \dots, m$.
(iii) There exists a threshold τ such that, whenever $X^e > \tau$, the representative players of active group i are players ik such that $\alpha_{ik} \geq \alpha_{il}$ for all $l = 1, \dots, m$.

Property (i) is due the fact that $s_i(X) \in (0, 1)$ for all $X \in (0, T_i)$. If there exists $X^e > 0$ such that $\sum_{i=1}^n s_i(X^e) = 1$, then there must be two distinct groups i and j such that both $s_i(X^e)$ and $s_j(X^e)$ are strictly positive. Property (ii) follows directly from the definition of T_{ik}. If $s_i(X^e) > 0$, then there exists a player ik such that $s_{ik}(X^e) = s_i(X^e)$ and $T_{ik} > X^e$. If $s_i(X^e) = 0$, then $s_{ik}(X^e) = 0$ and $T_{ik} \leq X^e$ for all $k = 1, \dots, m$. Property (iii) is a result of the fact that T_{ik} is strictly increasing in α_{ik}. Suppose that $\alpha_{ik} \geq \alpha_{il}$ for all $l = 1, \dots, m$. Then, there exists $\tau \in (0, T_{ik})$ such that $s_{ik}(X) > 0$ and $s_{ik}(X) \geq s_{il}(X)$ for all $X \in (\tau, T_{ik})$ and $l = 1, \dots, m$. If

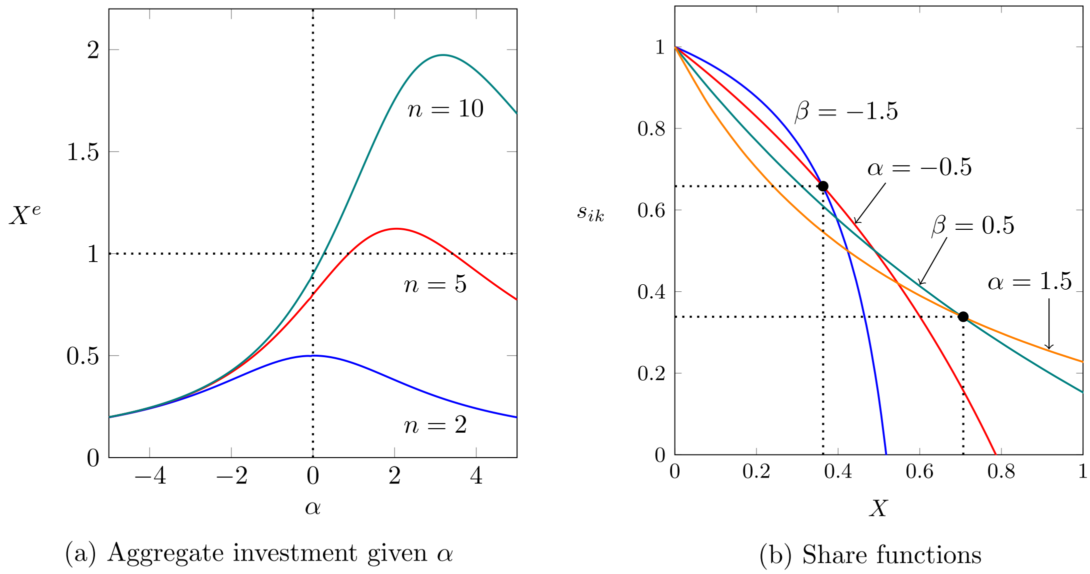


Fig. 2. Aggregate investment in a symmetric equilibrium.

$X^e \in (\tau, T_{ik})$, then group i is active and player ik is a representative player of group i.

Under the assumption of homogeneous players, i.e., $\alpha_{ik} = \alpha$ for all $i = 1, \ldots, n$ and $k = 1, \ldots, m$, we can explicitly derive the aggregate investment in equilibrium as a function of α, c, n, and V.

Proposition 2. *Suppose that $\alpha_{ik} = \alpha$ for all $i = 1, \ldots, n$ and $k = 1, \ldots, m$. Then, under Assumption 1, all groups are active in equilibrium, all players are representative, and*

$$X^e = \begin{cases} \frac{(n-1)V}{nc} & \text{if } \alpha = 0 \\ \frac{(n-1)(e^{\alpha V}-1)}{(n+e^{\alpha V}-1)\alpha c} & \text{if } \alpha \neq 0. \end{cases} \tag{15}$$

Given $n \geq 2$, there exists a unique value of α that maximizes X^e. We call such a value α^* and find that

(i) $\alpha^* = 0$ whenever $n = 2$, and
(ii) α^* is strictly increasing in n.

Proof. See Appendix A.

According to (15), we find that X^e is strictly decreasing in c and strictly increasing in n and V. The effect of α on X^e is nonmonotonic as illustrated in Fig. 2(a). Indeed, X^e is single-peaked in α, and the maximizing level is strictly increasing in n. This pattern is consistent with the contest between individuals case, and can be attributed to an increase in downside risk aversion.[13] For $\alpha < 0$, players become less risk averse and also less downside risk averse as α increases. As a result, each invests more in equilibrium. For $\alpha > 0$, as α increases, the players become less risk averse but more downside risk averse. The aggregate investment will begin to decline when α is large enough so that the positive effect of a decrease in risk aversion on investment is dominated by the negative effect of increased aversion to downside risk. Similar to the contest between individuals case, the effect of risk aversion on behavior is generally ambiguous (see Jindapon and Yang, 2017 for more details).

[13] For the utility function in (16), Modica and Scarsini (2005)'s absolute coefficient of downside risk aversion is $u'''_{ik}(w)/u'_{ik}(w) = \alpha^2_{ik}$. That is, as α increases, players become less risk averse and, at the same time, an increase in $|\alpha|$ increases players' downside risk aversion.

Part (ii) of Proposition 2 suggests some interesting design choices for a designer faced with distributing a fixed number of players into competing groups with equal size. Holding α fixed, Fig. 2(a) illustrates that X^e is monotonically increasing in the number of groups n. For example, if there are 10 players with the same α, organizing a contest with $(n, m) = (10, 1)$ can generate a larger aggregate investment than contests with $(5, 2)$ and $(2, 5)$ for any given α.

Corollary 2. *Suppose that $\alpha_{ik} = \alpha$ for all $i = 1, \ldots, n$ and $k = 1, \ldots, m$, and assume the total number of contestants is fixed $N = nm$. Then, under Assumption 1, X^e is maximized when $n^* = N$ and $m^* = 1$, i.e., in the contest between individuals.*

In fact, the symmetric equilibrium derived above can be obtained even when players are not homogeneous. Consider a contest where there is symmetry only among the least risk-averse players of all groups, i.e., $\max\{\alpha_{i1}, \ldots, \alpha_{im}\} = \alpha$ for all $i = 1, \ldots, n$. We already know that group i's share function $s_i(X)$ is given by (13) and we call all players such that $s_{ik}(X) = s_i(X)$ the representative players of group i given X. Thus, player ik is a representative player of group i in equilibrium if her share function is at least as high as all other players' share functions given $X = X^e$. We show in the following proposition that, when $\alpha \leq 0$, only players such that $\alpha_{ik} = \alpha$, i.e., the least risk-averse players of each group, are representative. Formally, suppose that β is the second largest value of α_{ik} in group i. Since $\alpha > \beta$, (9) implies that $T_\alpha > T_\beta$. It follows that there exists $\psi \in (0, T_\alpha)$ such that $s_\alpha(\psi) = s_\beta(\psi)$, $s_\alpha(X) > s_\beta(X)$ for all $X \in (\psi, T_\alpha)$, and $s_\alpha(X) < s_\beta(X)$ for all $X \in (0, \psi)$. In other words, ψ is the value of X where the two share functions cross. We find that if $\alpha \leq 0$, then $s_\alpha(\psi) = s_\beta(\psi) > \frac{1}{2}$. Since X^e is X such that $s_i(X) = \frac{1}{n}$, then $X^e > \psi$ for all $n \geq 2$ and, therefore, the players with $\alpha_{ik} = \alpha$ must be representative in equilibrium. See an example where $\alpha = -0.5$ and $\beta = -1.5$ in Fig. 2(b).

However, when $\alpha > 0$, the least risk-averse players (i.e., the most risk-loving players) in each group may not be representative. This is due to the possibility that $s_\alpha(\psi) = s_\beta(\psi) < \frac{1}{2}$ and X^e can be smaller than ψ. For the least risk-averse players to be representative, n must be so large that $s_\alpha(\psi) = s_\beta(\psi) > \frac{1}{n}$ and hence $X^e > \psi$. See another example where $\alpha = 1.5$ and $\beta = 0.5$ in Fig. 2(b). Since $s_\alpha(0.703) = s_\beta(0.703) = 0.339$, the representative players are those with $\alpha_{ik} = 1.5$ if and only if $n \geq 3$.

Proposition 3. *Suppose that Assumption 1 holds and that* $\max\{\alpha_{i1}, \ldots, \alpha_{im}\} = \alpha$ *for all* $i = 1, \ldots, n$.

(i) All groups are active in equilibrium.
(ii) If (a) $\alpha \leq 0$ *or (b)* $\alpha > 0$ *and*

$$n > \begin{cases} \frac{\alpha V(e^{\alpha V}-1)}{(e^{\alpha V}-1)-\alpha V} & \text{if } \beta = 0 \\ \frac{(\alpha-\beta)(e^{\alpha V}-1)(e^{\beta V}-1)}{\beta(e^{\alpha V}-1)-\alpha(e^{\beta V}-1)} & \text{if } \beta \neq 0, \end{cases} \tag{16}$$

where β *is the second largest value of* α_{ik}*, then players with* $\alpha_{ik} = \alpha$ *are the representative players of group* i*, and* X^e *is given by (15).*
(iii) If either (a) or (b) holds and there is only one player in group i *such that* $\alpha_{ik} = \alpha$*, then* $R_i(X^e)$ *is singleton and such a player is the only active player of group* i*.*

Proof. See Appendix B.

3.2. Convex cost

Now we assume that $g(x)$ is strictly convex.

Assumption 2. (i) $g(0) = g'(0) = 0$; (ii) $g'(x) > 0$ and $g''(x) > 0$ for all $x > 0$; (iii) $g'''(x)$ exists and is finite for all $x > 0$.

Most parts of Assumption 2 are standard in the literature; however, part *(iii)* is only needed for the quadratic approximation in Section 4.3.

If $\alpha_{ik} = 0$, the first-order condition for player ik is given by

$$\frac{X - X_i}{X^2} V - g'(x_{ik}) = 0. \tag{17}$$

If $\alpha_{ik} \neq 0$, the first-order condition for player ik is given by

$$\frac{X - X_i}{X^2}\left(\frac{e^{\alpha_{ik}W_{ik}^V} - 1}{\alpha_{ik}} - \frac{e^{\alpha_{ik}W_{ik}^L} - 1}{\alpha_{ik}}\right) - g'(x_{ik})\left(\frac{X_i}{X}e^{\alpha_{ik}W_{ik}^V} + \frac{X - X_i}{X}e^{\alpha_{ik}W_{ik}^L}\right) = 0, \tag{18}$$

where $W_{ik}^V = I + V - g(x_{ik})$ and $W_{ik}^L = I - g(x_{ik})$. We define w_{ik} as player ik's share of her contribution within group i so that $x_{ik} = w_{ik}X_i$. Since $X_i = s_iX$, both (17) and (18) can be written as

$$F_{ik}(w_{ik}, s_i|X) = 0 \tag{19}$$

where we define

$$F_{ik}(w_{ik}, s_i|X) := \begin{cases} (1 - s_i)V - g'(w_{ik}s_iX)X & \text{if } \alpha_{ik} = 0 \\ \frac{(1-s_i)(e^{\alpha_{ik}V}-1)}{(1-s_i+s_ie^{\alpha_{ik}V})\alpha_{ik}} - g'(w_{ik}s_iX)X & \text{if } \alpha_{ik} \neq 0 \end{cases} \tag{20}$$

given $X > 0$. We derive other important properties of $s_i(X)$ in Lemma 2.

Lemma 2. *Under Assumption 2, group i's share function,* $s_i(X)$*, is a continuous function given* $X > 0$ *with the following properties.*

(i) $s_i(X)$ *is strictly decreasing in* X *for all* $X > 0$.
(ii) $\lim_{X\to 0} s_i(X) = 1$.
(iii) $\lim_{X\to\infty} s_i(X) = 0$.

Proof. See Appendix C.

Proposition 4. *Under Assumption 2, there exists a pure strategy Nash equilibrium. The aggregate investment in equilibrium,* X^e*, is unique. Every player is active in equilibrium.*

Proof. Given the properties of $s_i(X)$ in Lemma 2, we find that $s(X) := \sum_{i=1}^n s_i(X)$ has the following properties: (i) $s(X)$ is strictly decreasing in X given $X > 0$, (ii) $\lim_{X\to 0} s(X) = n$; and (iii) $\lim_{X\to\infty} s(X) = 0$. Thus, there exists a unique value of X such that $s(X) = 1$. Since $g'(0) = 0$, the first order condition in (18) suggests that $x_{ik}^e > 0$. ■

4. Sorting

In this section, we explore how sorting – exogenously assigning a fixed pool heterogeneous players into competing groups – affects aggregate investment. We focus on two sortings. Under a *balanced* sorting, players are assigned to groups so as to minimize the variance in risk attitudes across groups. In contrast, an *unbalanced* sorting of players maximizes the variance in risk attitudes across groups.[14] In order to compare the two sortings for a given set of players, we assume a compatible set of preference parameters described below.

Assumption 3. There are n^2 players. The contest organizer assigns players to n groups so that each group has n members. Let $\alpha_{ik} = \alpha_{jk}$ for all i, j, k in the balanced sorting, and $\alpha_{ik} = \alpha_{il}$ for all i, k, l in the unbalanced sorting.

There is no between-group heterogeneity in the balanced sorting and there is no within-group heterogeneity in the unbalanced sorting. For example, in a group contest with 9 players, suppose that there are 3 different values of risk parameters and that exactly 3 players share the same value. In the balanced sorting, we assign the players into 3 identical groups and each group has 3 different players. In the unbalanced sorting, we assign identical players to the same group so that all 3 groups are different.

4.1. Linear cost

First we assume that each player's cost function is linear. In the following examples, we calculate an equilibrium for the unbalanced and balanced sorting in a group contest with 9 players.

Example 1 (*Unbalanced Sorting*)**.** Suppose that $n = 3$, $c = 1$, and $\alpha_{1k} = -1.5$, $\alpha_{2k} = -1$, and $\alpha_{3k} = -0.5$ for all k. We find that in equilibrium, $X^e = 0.5039$, $s_1^e = 0.1105$, $s_2^e = 0.4088$, and $s_3^e = 0.4807$. See Fig. 3(a).

Example 2 (*Balanced Sorting*)**.** Suppose that $n = 3$, $c = 1$, and $\alpha_{i1} = -1.5$, $\alpha_{i2} = -1$, and $\alpha_{i3} = -0.5$ for all i. We find that in equilibrium, $X^e = 0.6038$ and $s_i^e = \frac{1}{3}$ for all i. See Fig. 3(b).

In Example 1 (unbalanced sorting), players are sorted so that they are identical within each group. Since $s_{i1}(X) = s_{i2}(X) = s_{i3}(X) = s_i(X)$, for all i, all players are representative members of their group. We find that the aggregate investment in equilibrium is 0.5039 and the corresponding probability of winning for group 1, 2, and 3 are 0.1105, 0.4088, and 0.4807, respectively. Group 1 invests $0.1105 \times 0.5039 = 0.0557$ in the contest. Since everyone in group 1 is a representative member, any vector $\mathbf{x}_1^e$ such that $x_{11}^e + x_{12}^e + x_{13}^e = 0.0557$ constitutes the equilibrium. We can derive vectors $\mathbf{x}_2^e$ and $\mathbf{x}_3^e$ in the same manner.

In Example 2 (balanced sorting), players are sorted so that all 3 groups are identical. The 3 players in each group have a different risk preference parameter. In Fig. 3(b), we plot $s_{ik}(X)$ using a dashed curve and emphasize the maximum value of $s_{ik}(X)$

[14] For consistency, we used the definitions of the balanced and unbalanced sorting from earlier literature (see, e.g., Ryvkin, 2011; Brookins et al., 2015b). In Section 4.3, we demonstrate that these two sortings have the most prominent impact on aggregate effort. However, there are $\frac{(mn)!}{(m!)^n n!}$ sorting possibilities.

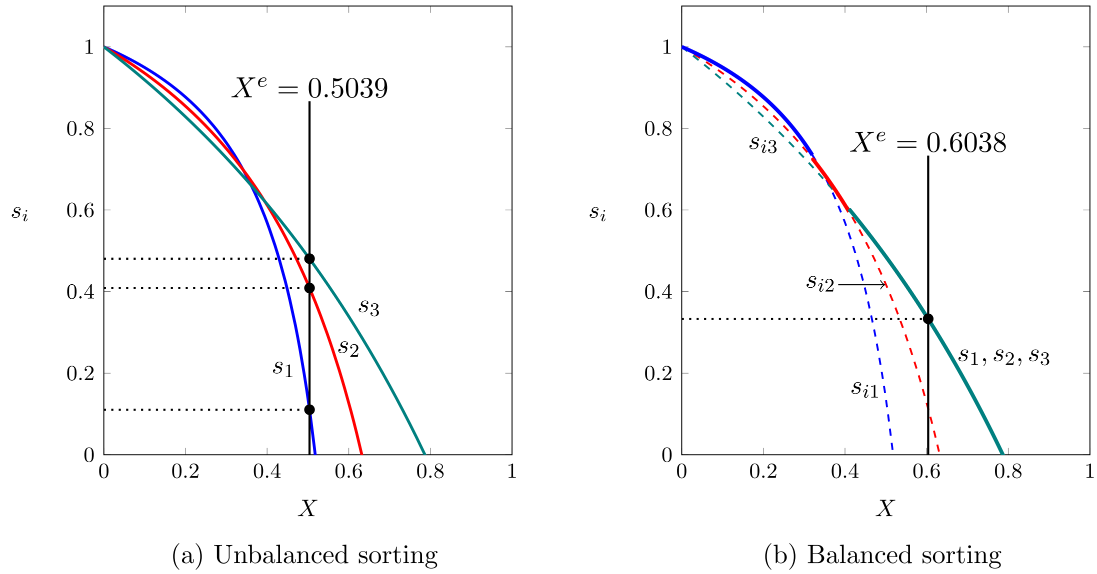


Fig. 3. Share functions given linear cost of investment in Examples 1 and 2.

in bold. Thus, according to (13), the bold plot is group i's share function $s_i(X)$. Given 3 identical groups, the aggregate investment in equilibrium is X such that $s_i(x) = \frac{1}{3}$. We find that $X^e = 0.6038$ and $X_i^e = 0.2013$ for all i. Given $X^e = 0.6038$, player $i3$ (the least risk-averse player in group i) is the only representative member of group i. Therefore $x_{i3}^e = 0.2013$ and $x_1^e = x_2^e = 0$ for all i.

According to these examples, we find that the aggregate investment in equilibrium is higher in the balanced sorting than the unbalanced sorting. In the next proposition, we prove that this finding (weakly) holds in general.[15]

Proposition 5. *Under Assumptions 1 and 3, the balanced sorting yields a (weakly) higher aggregate investment and a (weakly) larger number of active groups than the unbalanced sorting.*

Proof. Let $s_i^b(X)$ and $s_i^u(X)$ be group i's share function under the balanced and unbalanced sorting respectively. We find that $s_i^b(X) = \max\{s_{i1}(X), \ldots, s_{in}(X)\}$ and $s_i^u(X) = s_{i1}(X) = \cdots = s_{in}(X)$. Thus, $\sum_{i=1}^n s_i^b(X) \geq \sum_{i=1}^n s_i^u(X)$ for all $X \in (0, T)$. Since both $\sum_{i=1}^n s_i^b(X)$ and $\sum_{i=1}^n s_i^u(X)$ are strictly decreasing in X, we find that $X^b \geq X^u$ where $\sum_{i=1}^n s_i^b(X^b) = \sum_{i=1}^n s_i^u(X^u) = 1$. Under a balanced sorting, $\alpha = \max\{\alpha_{i1}, \ldots, \alpha_{in}\}$ for all $i = 1, \ldots, n$. Thus, Proposition 3 implies $n_b = n$, and hence, $n_b \geq n_u$. ■

Another important finding from Example 2 (balanced sorting) is that the least risk-averse player in each group, i.e., player $i3$, is the only representative player of group i, while the more risk-averse players, i.e., players $i1$ and $i2$, free ride (see Proposition 3 part (*ii*) case (*a*) and part (*iii*)). However, if some of the players are risk loving, then the least risk-averse player (i.e., the most risk-loving player) in each group will be the representative players provided the number of groups is sufficiently large (see Proposition 3 part (*ii*) case (*b*)). In the following example, we show that, under a balanced sorting, the least risk-averse players are free riders when $n = 3$, but they are the representative players when $n = 6$.

Example 3 (*Balanced Sorting*)**.** $n = 3$ and $c = 1$. We let $\alpha_{i1} = -0.5$, $\alpha_{i2} = 1$, and $\alpha_{i3} = 2.5$ for all i. We find in equilibrium, $X^e = 0.7283$, $s_i^e = \frac{1}{3}$, and $X_i^e = 0.2428$ for all i. For each player, $x_{i1}^e = x_{i3}^e = 0$ and $x_{i2}^e = 0.2428$ for all i. See Fig. 4(a). Now suppose that $n = 6$ with $\alpha_{i1} = \alpha_{i2} = -0.5$, $\alpha_{i3} = \alpha_{i4} = 1$, and $\alpha_{i5} = \alpha_{i6} = 2.5$ for all i. We find that $X^e = 1.3016$, $s_i^e = \frac{1}{6}$, and $X_i^e = 0.2169$ for all i. For each player, $x_{i1}^e = \cdots = x_{i4}^e = 0$ and $x_{i5}^e = x_{i6}^e = 0.1085$ for all i. See Fig. 4(b).

4.2. Convex cost

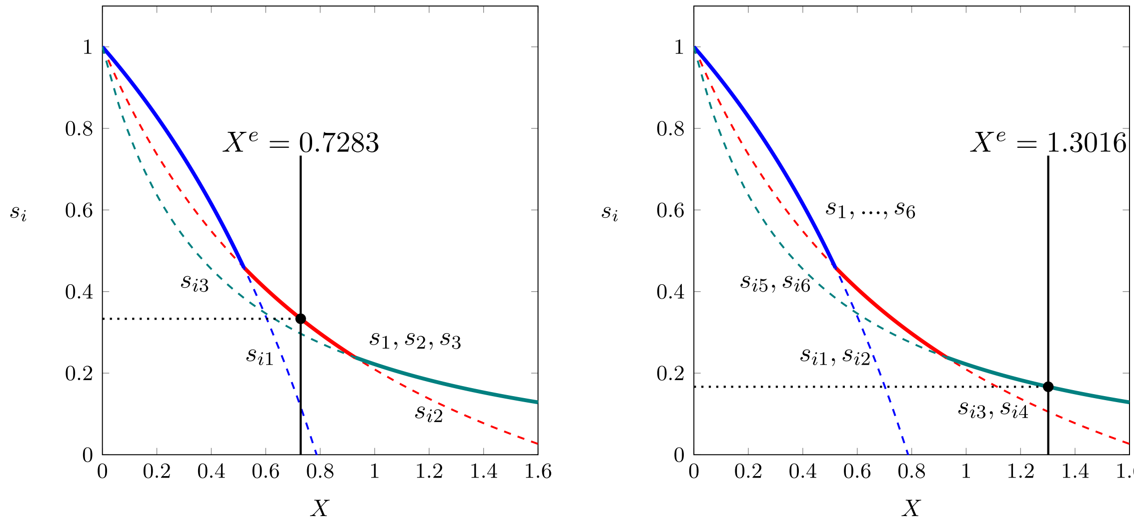
Now suppose that each player's cost function is convex as described in Assumption 2. In Examples 4 and 5, we adopt the same set of players as in Assumption 3 and the same sorting method as in Examples 1 and 2, but assume the cost function $g(x_{ik}) = x_{ik}^2$.

Example 4 (*Unbalanced Sorting*)**.** $n = 3$ and $g(x_{ik}) = x_{ik}^2$ for all i, k. We let $\alpha_{1k} = -1.5$, $\alpha_{2k} = -1$, and $\alpha_{3k} = -0.5$, for all k. We find in equilibrium, $X^e = 1.5437$, $s_1^e = 0.2978$, $s_2^e = 0.3356$, and $s_3^e = 0.3666$. See Fig. 5(a). It follows that $X_1^e = 0.4597$, $X_2^e = 0.5181$, and $X_3^e = 0.5659$. Since $w_{ik}^e = \frac{1}{3}$ for all i and k, $\mathbf{x}_1^e = (0.1532, 0.1532, 0.1532)$, $\mathbf{x}_2^e = (0.1727, 0.1727, 0.1727)$, and $\mathbf{x}_3^e = (0.1886, 0.1886, 0.1886)$.

Example 5 (*Balanced Sorting*)**.** $n = 3$ and $g(x_{ik}) = x_{ik}^2$ for all i, k. We let $\alpha_{i1} = -1.5$, $\alpha_{i2} = -1$, and $\alpha_{i3} = -0.5$, for all i. We find in equilibrium, $X^e = 1.5510$ and $s_i^e = \frac{1}{3}$ for all i. See Fig. 5(b). It follows that $X_i^e = 0.5170$ and $w_{i1}^e = 0.2905$, $w_{i2}^e = 0.3329$, and $w_{i3}^e = 0.3765$ for all i. Thus, $\mathbf{x}_i^e = (0.1502, 0.1721, 0.1947)$ for all i.

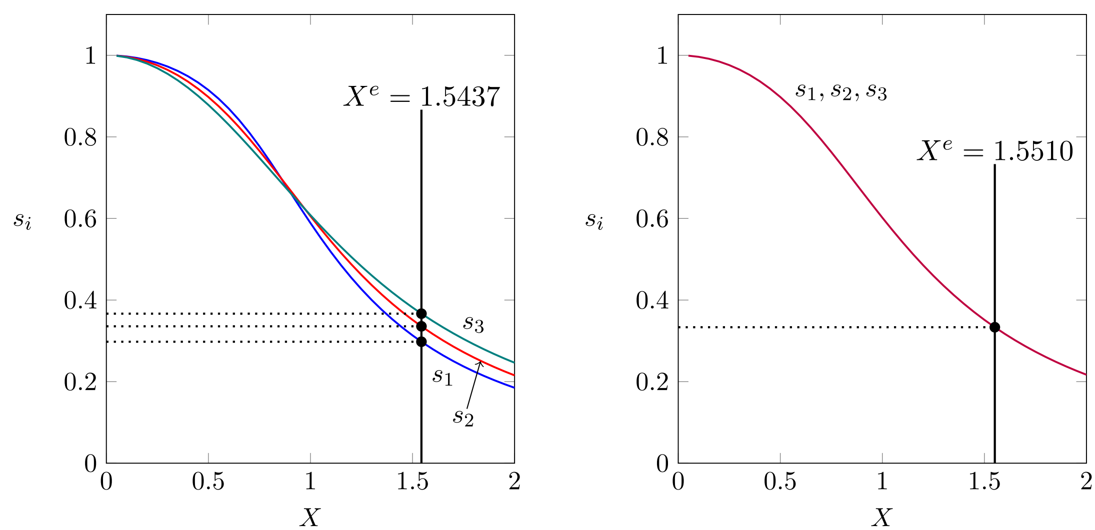
Even though we find in Examples 4 and 5 that the balanced sorting yields a higher aggregate investment than the unbalanced sorting, this finding does not hold in general. Unlike in the linear costs case, we are unable to prove a statement akin to Proposition 5 through share functions when costs are convex. However, our exploratory numerical results indicated by Table 1 in tandem with Fig. 6 indicate some interesting relationships between aggregate investment, the average risk preference across

[15] While the balanced sorting yields a strictly higher aggregate investment than the unbalanced sorting given *almost all* sets of players, there are some special cases in which the aggregate investments in equilibrium are the same under both sortings. For example, suppose that there are four players assigned to two groups of two, and that the risk preference parameters are α for two of the players and $-\alpha$ for the other two. Since $s_i(X^e) = s_{ik}(X^e) = \frac{1}{2}$ for all i, k given any sorting, the probability of winning in equilibrium is $\frac{1}{2}$ for both groups and the aggregate investment in equilibrium is the same given any sorting.



(a) $n = 3$ (b) $n = 6$

Fig. 4. Share functions given the balanced sorting in Example 3.



(a) Unbalanced sorting (b) Balanced sorting

Fig. 5. Share functions given convex cost of investment in Examples 4 and 5.

players, and the degree of heterogeneity. In Table 1 and Fig. 6, we describe each player's risk parameter by $\alpha_{ik} = \bar{\alpha}(1 + a_{ik})$, where $\bar{\alpha}$ is the average value of α_{ik} and a_{ik} represents player ik's relative deviation from $\bar{\alpha}$. Given $n = 3$ and $g(x_{ik}) = x_{ik}^2$, we find that (i) for $\bar{\alpha} < 0$, X^e is higher in the balanced sorting compared to the unbalanced sorting, (ii) for $\bar{\alpha} > 0$, there exists a range $[\bar{\alpha}_L, \bar{\alpha}_H]$ such that X^e is higher in the unbalanced sorting compared to the balanced sorting, but otherwise X^e is higher in the balanced sorting, (iii) for a given spread in risk attitudes, X^e is nonmonotonic in $\bar{\alpha}$, and (iv) for a given $\bar{\alpha}$, X^e is typically, but not always, decreasing in heterogeneity parameter a_{ik}. To demonstrate findings (i) and (ii) further, Fig. 7 plots the difference in aggregate investment X^e between sortings (balanced minus unbalanced) as a function of $\bar{\alpha}$. As depicted, for various parameters a_{ik}, there exists a range $[\bar{\alpha}_L, \bar{\alpha}_H]$ such that for $\bar{\alpha} \in [\bar{\alpha}_L, \bar{\alpha}_H]$ the unbalanced sorting produces higher aggregate investment than the balanced sorting; otherwise, the balanced sorting is optimal. In Section 4.3, we show that (i)–(iv) do not generally hold, and shed more light on the effects of heterogeneity and sorting on X^e via an explicit approximation.

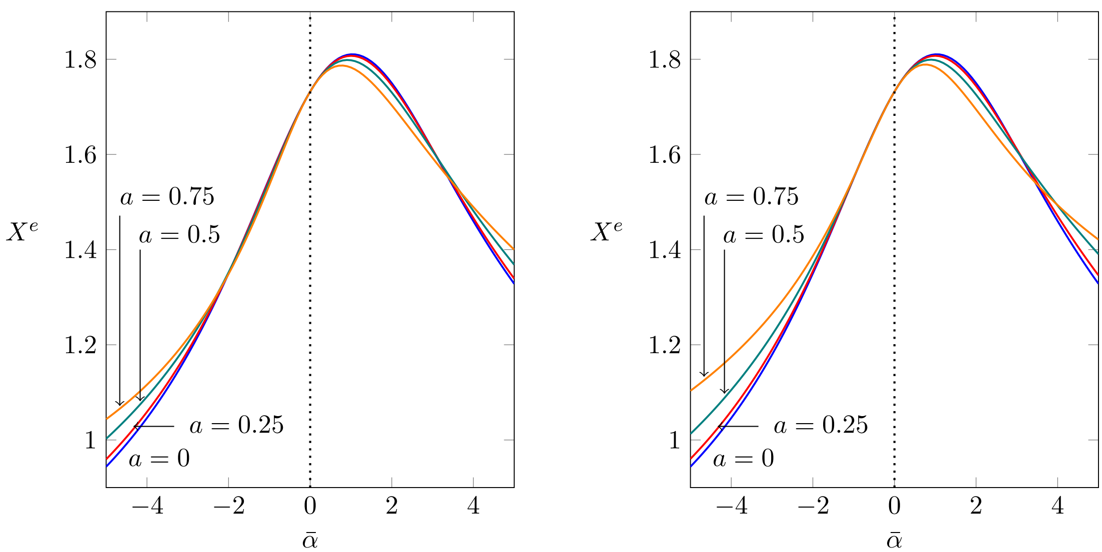
4.3. Sorting with weakly heterogeneous players

To obtain an explicit approximation of X^e, we assume the heterogeneity present in the model is *weak* and use perturbation analysis to approximate an explicit expression for the deviation in aggregate investment from the symmetric equilibrium level.[16] First, consider the group contest where players have homogeneous risk attitudes, $\alpha_{ik} = \bar{\alpha}$ for all i, k. The unique symmetric equilibrium investment level $\bar{x}$ satisfies first-order condition (18)

16 In contest models with risk-averse players, obtaining an explicit expression for equilibrium strategies is typically not possible. Several papers have used perturbation analysis to circumvent this issue. For example, Fibich et al. (2006) use a similar approach to analyze all-pay auctions with symmetrically risk-averse players.

Table 1
A comparison of aggregate investment X^e across the balanced and unbalanced sortings given $n = 3$ and $g(x_{ik}) = x_{ik}^2$.

	Deviation of α_{ik} from $\bar{\alpha}$ represented by the values of a_{ik}			
	$a_{ik} = 0$ for all ik	$\{-0.25, 0, 0.25\}$	$\{-0.50, 0, 0.50\}$	$\{-0.75, 0, 0.75\}$
$\bar{\alpha} = -4$				
⋆ Unbalanced	1.0461	1.0590	1.0908	1.1161
⋆ Balanced	1.0461	1.0608	1.1051	1.1746
$\bar{\alpha} = -2$				
⋆ Unbalanced	1.3499	1.3510	1.3522	1.3487
⋆ Balanced	1.3499	1.3547	1.3682	1.3866
$\bar{\alpha} = 2$				
⋆ Unbalanced	1.7499	1.7452	1.7302	1.7031
⋆ Balanced	1.7499	1.7439	1.7255	1.6945
$\bar{\alpha} = 4$				
⋆ Unbalanced	1.4597	1.4662	1.4813	1.4894
⋆ Balanced	1.4597	1.4705	1.4924	1.4934



(a) Unbalanced sorting (b) Balanced sorting

Fig. 6. A comparison of aggregate investment X^e given $g(x_{ik}) = x_{ik}^2$ and $n = 3$. In the balanced sorting, players are sorted into groups by risk attitudes, $\alpha_{ik} = \bar{\alpha}(1+a_{ik})$, as follows: $a_{i1} = a$, $a_{i2} = 0$ and $a_{i3} = -a$ for $i = 1, 2, 3$, and in the unbalanced sorting: $a_{1k} = a$, $a_{2k} = 0$ and $a_{3k} = -a$ for $k = 1, 2, 3$.

with equality for all players ik, and hence, satisfies

$$\frac{(n-1)}{n^2 m \bar{x}} \bar{\beta} = \bar{\alpha} g'(\bar{x}) \left(\frac{1}{n} \bar{\beta} + 1 \right), \tag{21}$$

where $\bar{\beta} := e^{\bar{\alpha} V} - 1$.

In contrast, assume now that players are heterogeneous in their absolute degree of risk aversion.[17] Like the numerical examples in Table 1, we present preference heterogeneity as a deviation from the average level $\bar{\alpha}$. Specifically, player ik's risk parameter can be written as $\alpha_{ik} = \bar{\alpha}(1 + a_{ik})$. Without loss of generality, we define the average level of risk aversion as $\bar{\alpha} := (nm)^{-1} \sum_{i,k} \alpha_{ik}$, implying parameters a_{ik} are centered symmetrically about zero, and $A := \sum_{i,k} a_{ik} = 0$. Let $S_a^2 := (nm)^{-1} \sum_{i,k} a_{ik}^2$ denote the sample variance of risk parameters across players, and denote the sample variance of risk parameters across groups by $S_A^2 := (n)^{-1} \sum_i A_i^2$, where $A_i := \sum_k a_{ik}$ is the aggregate risk parameter of group i. A key assumption in this section is that of weak heterogeneity. Letting $\rho = \max_{ik}|a_{ik}|$, the assumption of weak heterogeneity implies that $\rho \ll 1$, while $\rho < 1$ implies that α_{ik} has the same sign as $\bar{\alpha}$. Although this assumption states that the deviations from the average risk parameter are small compared to unity, the quadratic approximation holds exceptionally well for higher spreads in the players' heterogeneity.

Our goal is to find approximate equilibrium investments in the group contest with heterogeneous players. By Proposition 4, equilibrium investment levels are the solutions to the system of equality equations (18). As with the risk parameters, we write equilibrium investment as $x_{ik}^e := \bar{x}(1 - b_{ik})$, where b_{ik} represents a small relative deviation from the symmetric equilibrium level for player ik. Thus, we expect that increases in risk parameters relative to the symmetric level, i.e., $a_{ik} > 1$, will lead to decreases in equilibrium investment relative to the symmetric level, i.e., $b_{ik} > 0$, in such a way that deviations b_{ik} will also be small compared to unity.

The quadratic approximation is obtained in four steps. First, we assume the solutions to the system (18) with $x_{ik}^e = \bar{x}(1 - b_{ik})$ and $\alpha_{ik} = \bar{\alpha}(1 + a_{ik})$ can be Taylor-expanded in powers of ρ, which in this case requires finding a Taylor expansion of relative investment deviations b_{ik} to at least the second order.[18]

[17] Assumption 3 is not necessary for our approximation results in this section. We allow for m players in each group, where m can be different from n, and also allow for a more general set of risk parameters.

[18] As we show in Appendix D, a second-order approximation is needed because the first-order term for aggregate relative investment deviation $B^{(1)} = 0$. Thus, to explore the effects of heterogeneity on aggregate output, we must (at least) obtain $B^{(2)}$.

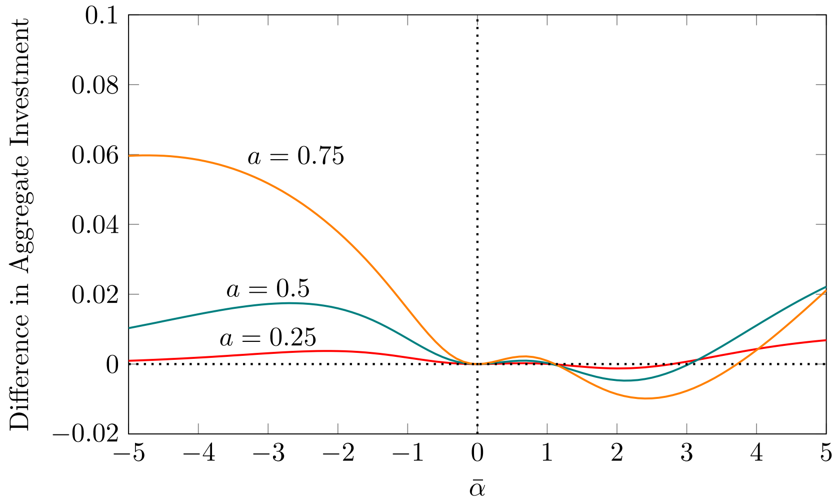


Fig. 7. Difference in aggregate investment X^e (balanced minus unbalanced) given $g(x_{ik}) = x_{ik}^2$ and $n = 3$. In the balanced sorting, players are sorted into groups by risk attitudes, $\alpha_{ik} = \bar{\alpha}(1 + a_{ik})$, as follows: $a_{i1} = a$, $a_{i2} = 0$ and $a_{i3} = -a$ for $i = 1, 2, 3$, and in the unbalanced sorting: $a_{1k} = a$, $a_{2k} = 0$ and $a_{3k} = -a$ for $k = 1, 2, 3$.

The approximated relative investment deviation of player ik is $b_{ik} := b_{ik}^{(1)} + b_{ik}^{(2)} + O(\rho^3)$, where $b_{ik}^{(s)}$ denotes the sth order term in the Taylor expansion of b_{ik} in powers of ρ, and $O(\rho^3)$ the approximation error. Similarly, write group and aggregate relative investment deviations as $B_i := B_i^{(1)} + B_i^{(2)} + O(\rho^3)$ and $B := B^{(1)} + B^{(2)} + O(\rho^3)$, respectively. Second, both sides of redefined equality (18) can be Taylor-expanded to the second order in powers of ρ. Third, successively sum all $O(\rho^s)$ terms and equate to zero, for $s = 0, 1, 2$, and fourth, successively solve each equation to obtain B, B_i and b_{ik}.

Proposition 6. *Under Assumption 2, the following holds in the quadratic approximation:*
(i) Aggregate equilibrium investment is

$$X^e = \bar{x}(nm - B^{(2)}) + O(S_a^2), \tag{22}$$

where

$$B^{(2)} = \lambda_a S_a^2 + \lambda_A S_A^2, \tag{23}$$

with λ_a and λ_A as given in Box I.
(ii) If $\lambda_A < 0$ ($\lambda_A > 0$), then X^e is maximized (minimized) when S_A^2 is maximized (minimized).
(iii) For a given $\bar{\alpha}$, $g(x_{ik})$, and sorting, X^e may be increasing or decreasing in S_a^2.

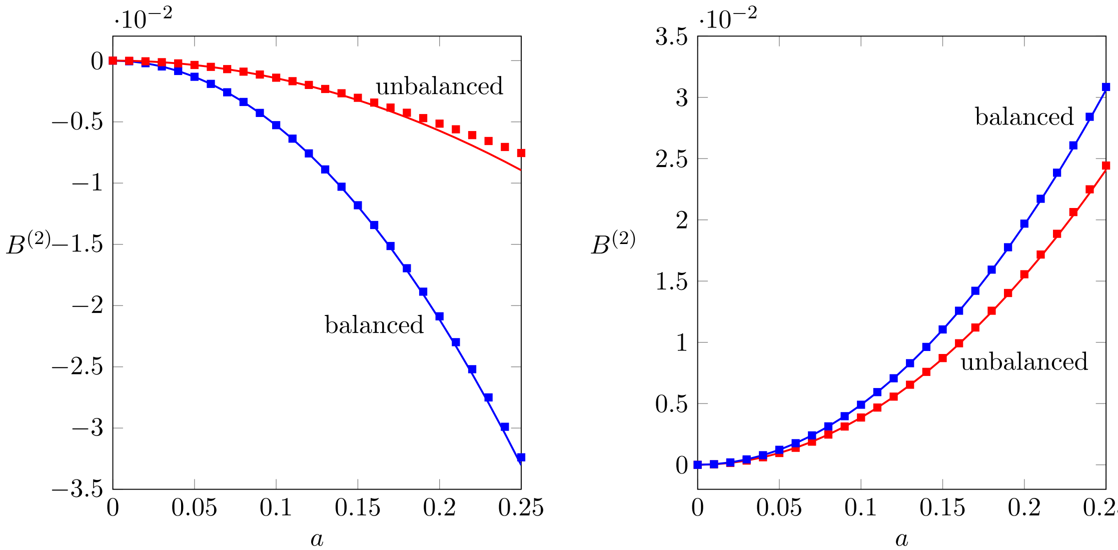
Proof. See Appendix D.

Part (i) demonstrates that the aggregate deviation from the symmetric equilibrium level can be written as a linear combination of the sample variance in risk parameters across players and across groups. For a given set of risk parameters, $\lambda_a S_a^2$ is constant. However, the sign and magnitude of $\lambda_A S_A^2$ can be altered. According to part (ii), if $\lambda_A < 0$, then aggregate investment is maximized when the sample variance across groups, S_A^2, is as large as possible, i.e., under an unbalanced sorting; the opposite holds when $\lambda_A > 0$. This is an interesting finding on its own, as it demonstrates that risk-based sorting matters and can be leveraged by a contest designer to achieve a desired goal. How is the sign of λ_A determined? Unfortunately, coefficient λ_A is a complicated function of model parameters, thereby making it difficult to determine its sign. In the numerical examples below, we show that the "steepness" of the convex investment cost plays a nontrivial role in determining the optimal sorting. That is, *ceteris paribus*, changing the cost function – and potentially changing the sign and/or magnitude of k_2 and k_3 – can reverse the dominance of one sorting over the other provided $\bar{\alpha} > 0$. Under risk neutrality, similar reversals have been found in group contest models with weak ability heterogeneity (Ryvkin, 2011; Brookins et al., 2015b).

Part (iii) states that aggregate investment may be decreasing or increasing in the degree of heterogeneity across players, and that the direction of the effect depends not only on the distribution of risk parameters, but also on the sorting and cost function. In the general group contest setting, determining the effect of heterogeneity on aggregate investment is challenging due to the simultaneous changes in individual and group-level variance in risk parameters. Such a challenge remains, but is mitigated when group-level variance in risk parameters is no longer relevant, i.e., in an individual contest. In Appendix E, we show that the effect of S_a^2 on X^e is ambiguous even when $m = 1$. When costs are sufficiently steep, X^e is decreasing in S_a^2. However, when costs are relatively less steep, X^e is increasing in S_a^2.

In Fig. 8, we present numerical illustrations to show the approximations fit and to identify some of the properties outlined in Proposition 6. In each panel, we assume parameters $g(x_{ik}) = x_{ik}^2$, $n = m = 3$. In the balanced sorting, players are sorted into groups by risk attitudes, $\alpha_{ik} = \bar{\alpha}(1 + a_{ik})$, as follows: $a_{i1} = a$, $a_{i2} = 0$ and $a_{i3} = -a$ for $i = 1, 2, 3$, and in the unbalanced sorting: $a_{1k} = a$, $a_{2k} = 0$ and $a_{3k} = -a$ for $k = 1, 2, 3$. Therefore, parameter $a \in [0, 0.25]$ controls the degree of heterogeneity across players. The quadratic approximation is given by the solid curves, and solid squares indicate numerical solutions. In Fig. 8(a), when the average risk parameter is $\bar{\alpha} = -2$, we see that $B^{(2)}$ is decreasing in a for each sorting, implying X^e is increasing in S_a^2. Also, notice that for all $a > 0$, $B^{(2)}$ is higher in the unbalanced sorting, and hence, X^e is always higher under the balanced sorting. In contrast, $B^{(2)}$ is increasing (i.e., X^e is decreasing) in a and higher under the balanced sorting for $a > 0$ in Fig. 8(b); therefore, the unbalanced sorting is optimal.

In Fig. 9, we show that the optimality of one sorting over the other can be reversed provided investment costs are sufficiently steep. In each panel, $n = m = 3$, risk attitudes are the same as in Fig. 8, and $\bar{\alpha} = 2.25$. In Fig. 9(a), investment costs are given by $g(x_{ik}) = x_{ik}^2$. Just as in Fig. 8(b), the unbalanced sorting yields higher aggregate investment than the balanced sorting for all $a > 0$. In Fig. 9(b), investment costs are given by a much



(a) $\bar{\alpha} = -2$ (b) $\bar{\alpha} = 2$

Fig. 8. Aggregate relative investment deviation, $B^{(2)}$, from the symmetric level under the balanced and unbalanced sorting as a function of the degree of heterogeneity, a, given $g(x_{ik}) = x_{ik}^2$ and $n = m = 3$. In the balanced sorting, players are sorted into groups by risk attitudes, $\alpha_{ik} = \bar{\alpha}(1 + a_{ik})$, as follows: $a_{i1} = a$, $a_{i2} = 0$ and $a_{i3} = -a$ for $i = 1, 2, 3$, and in the unbalanced sorting: $a_{1k} = a$, $a_{2k} = 0$ and $a_{3k} = -a$ for $k = 1, 2, 3$. The quadratic approximation is given by the solid curves, and solid squares indicate numerical solutions.

$$\lambda_a = \frac{mn\left\{\delta\left[k_3\delta + 2k_2\bar{\beta}(\bar{\beta}+n)\right] + k_2\bar{\beta}(\bar{\beta}+1)\ln(\bar{\beta}+1)\left[2(\delta+\bar{\beta}(\bar{\beta}+n)) - n(\bar{\beta}+n)\ln(1+\bar{\beta})\right]\right\}}{2k_2\bar{\beta}^2(1+k_2)(\bar{\beta}+n)^2},$$

$$\lambda_A = \frac{n\delta}{2k_2 m\bar{\beta}^2\gamma^2(1+k_2)(\bar{\beta}+n)^2}\left\{k_3 n\delta(n-2\gamma) - 2k_2\gamma\bar{\beta}^3 + 2n\bar{\beta}\left[k_2^2\delta(n-1)^2 - k_2 n\gamma + k_3\delta(n-\gamma)\right]\right.$$
$$\left. + \bar{\beta}^2\left[\delta(2k_2^2(n-1)^2 + k_3 n^2) - 4k_2 n\gamma\right] + 2k_2\gamma n^2(\bar{\beta}+1)^2\ln(\bar{\beta}+1)\right\},$$

$$\delta = n(\bar{\beta}+1)\ln(\bar{\beta}+1) - \bar{\beta}(\bar{\beta}+n),\ \gamma = n(\bar{\beta}+1) + k_2(n-1)(\bar{\beta}+n),\ k_2 = \frac{\bar{x}g''(\bar{x})}{g'(\bar{x})},\ k_3 = \frac{\bar{x}g'''(\bar{x})}{g''(\bar{x})}$$

Box I.

steeper function, $g(x_{ik}) = \frac{x_{ik}}{500}\int_{1-x_{ik}}^{1} t^{-1}\exp(t)dt$.[19] In this case, the optimal sorting is reversed: for all $a > 0$, X^e is higher under the balanced sorting than the unbalanced sorting. Moreover, as suggested by part (iii) of Proposition 6, X^e can be increasing or decreasing in S_a^2, depending on the sorting.

So far, we have allowed for any convex function satisfying Assumption 2. While we obtain the general prediction in Proposition 6, we trade-off tractability. We now turn to the quadratic cost function that we used in Section 4.2 to analyze the effect of between-group variance on the aggregate investment in equilibrium so we can identify the optimal sorting for this specific cost function. Under $g(x_{ik}) = x_{ik}^2$, $k_2 = 1$ and $k_3 = 0$, allowing us to simplify λ_A derived in Proposition 6 as in Box II, where $\gamma = n(\bar{\beta}+1)+(n-1)(\bar{\beta}+n) > 0$ and $\delta = n(\bar{\beta}+1)\ln(\bar{\beta}+1)-\bar{\beta}(\bar{\beta}+n) > 0$. For all $\bar{\alpha} < 0$, i.e., $\bar{\beta} \in (-1, 0)$, it can be easily verified that $\lambda_A > 0$ given any $n \geq 2$ and $m \geq 1$. This result confirms our previous numerical simulations presented in Fig. 7 in which the balanced sorting is always optimal when $\bar{\alpha} < 0$.

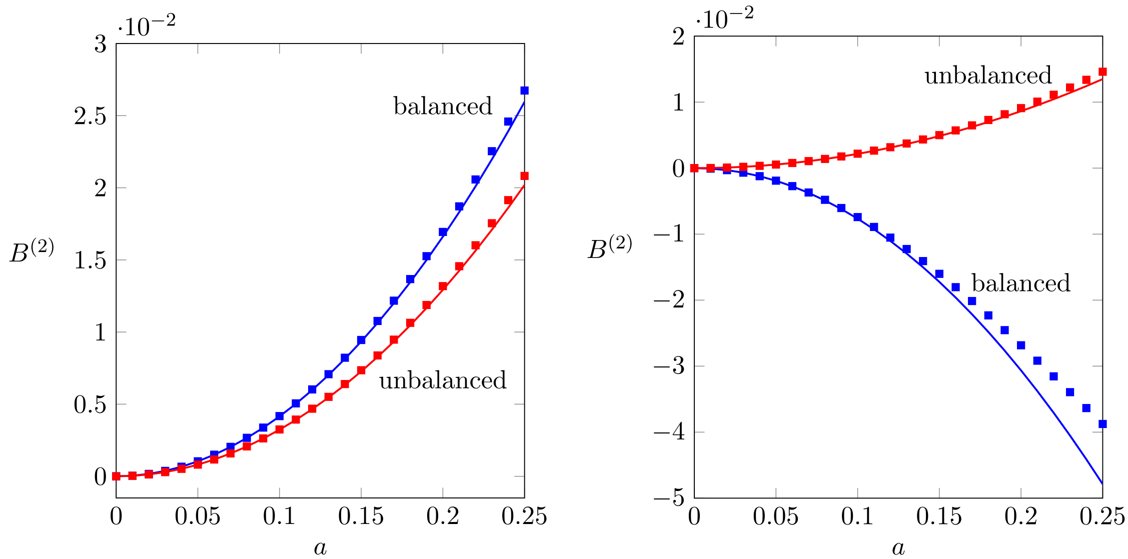
Corollary 3. *Suppose that $\bar{\alpha} < 0$ and $g(x_{ik}) = x_{ik}^2$. Then, $\lambda_A > 0$ and X^e is larger under the balanced sorting than the unbalanced sorting.*

[19] Note that $g(x_{ik}) = \frac{x_{ik}}{500}\int_{1-x_{ik}}^{1} t^{-1}\exp(t)dt = x_{ik}[\mathrm{Ei}(1) - \mathrm{Ei}(1-x_{ik})]/500$, where $\mathrm{Ei}(x)$ is the exponential integral function.

Finally, we acknowledge the performance of our quadratic approximation. In each of our numerical illustrations, the precision of the approximation relative to the numerical solution is quite good. In particular, it continues to perform well for heterogeneity up to, and in most cases beyond, $a_{ik} = 0.2$, which translates into a 20% relative difference from the symmetric level. Therefore, our approximate results provide a reliable depiction of the equilibrium under a wide range of parameter specifications.

5. Conclusion

In this paper, we presented and analyzed the first model of a group contest in which players could differ in their attitudes towards risk, i.e., players can be risk averse, risk neutral, or risk loving. We proved equilibrium existence and uniqueness in a class of imperfectly discriminating contests under both linear and convex costs. For the case of linear costs, we identify each group's representative members and find that only representative members are active in equilibrium. If all players are risk averse, then each group's representative members are the least risk-averse players in the group. However, if some players are risk loving, the most risk-loving players may not be representative members and thus free ride in equilibrium. In contrast, there are no free riders in the case of convex costs. Our results not only contribute to the literature studying group contests, but also to



(a) $g(x_{ik}) = x_{ik}^2$ (b) $g(x_{ik}) = \frac{x_{ik}}{500}\int_{1-x_{ik}}^{1} t^{-1}\exp(t)dt$

Fig. 9. Aggregate relative investment deviation, $B^{(2)}$, from the symmetric level under the balanced and unbalanced sorting as a function of the degree of heterogeneity, a, given $\bar{\alpha} = 2.25$ and $n = m = 3$. In the balanced sorting, players are sorted into groups by risk attitudes, $\alpha_{ik} = \bar{\alpha}(1 + a_{ik})$, as follows: $a_{i1} = a$, $a_{i2} = 0$ and $a_{i3} = -a$ for $i = 1, 2, 3$, and in the unbalanced sorting: $a_{1k} = a$, $a_{2k} = 0$ and $a_{3k} = -a$ for $k = 1, 2, 3$. The quadratic approximation is given by the solid curves, and solid squares indicate numerical solutions.

$$\lambda_A = \frac{n\delta\left\{-2\gamma\bar{\beta}^3 + 2n\bar{\beta}\left[\delta(n-1)^2 - n\gamma\right] + \bar{\beta}^2\left[2\delta(n-1)^2 - 4n\gamma\right] + 2\gamma n^2(\bar{\beta}+1)^2\ln(\bar{\beta}+1)\right\}}{4m\bar{\beta}^2\gamma^2(\bar{\beta}+n)^2}$$

Box II.

the literature exploring the effects of risk preferences on behavior in competitive situations, which up until now only consisted of a few studies.

Beyond our existence and uniqueness analyses, we focused our attention on one intriguing application of heterogeneity in group contests. We asked the following question from a contest designer's perspective: *If I want to maximize the sum of investments across players, how should I sort them by risk attitude into competing groups?* More specifically, should the contest designer create an unbalanced sorting, where she assigns more risk-averse individuals to one team, thereby assigning less risk-averse individuals to the other team? Or, should she create a balanced sorting, in which risk attitudes are assigned to teams so as to create two similar teams? Such a sorting decision is particularly relevant in organizational settings utilizing relative performance incentives (e.g., team sales competitions), because risk preference rankings can be directly inferred from psychometric testing and repeated interactions. In addition, risk preferences can be indirectly inferred from observable and correlated information such as age and sex. Under linear costs, we obtain a clear-cut answer that the balanced sorting produces higher aggregate investment than the unbalanced sorting. To investigate the robustness of this result, we additionally analyzed the optimal sorting under convex costs, and find that the optimal sorting is more nuanced. Specifically, the common wisdom that "competitive balance promotes output" does not generally hold.

Since this paper offers the first equilibrium analysis and investigation into the consequences of risk preference heterogeneity, there are still many extensions, both theoretically and empirically, to consider. In theory, one may assume a different group production function, other forms of heterogeneity, and consider a more general prize structure. While we assume a group-specific public good prize only for the winning group, one could analyze an equilibrium in public good funding lotteries, where all players (from winning and non-winning groups) could benefit from the public good provided by the contest organizer.[20] Prior experimental studies exploring optimal sorting in group contests with ability heterogeneity find that the balanced sorting is optimal, even in situations when the unbalanced sorting is theoretically optimal (Brookins et al., 2015a, 2018). Therefore, it would be interesting to empirically test the dominance of one sorting over the other in our setting.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We are grateful to Carmen Beviá, Jörg Franke, and three anonymous reviewers for helpful feedback on our manuscript.

Appendix A. Proof of Proposition 2

Given homogeneous players, $s_i(X) = s_{ik}(X)$ as in (11) if $\alpha = 0$ and (12) if $\alpha \neq 0$. Setting $s_i(X) = \frac{1}{n}$ yields (15). Given $\alpha \neq 0$, we

[20] See Bos (2011) for an analysis of prize and altruistic preference heterogeneity and Jindapon and Yang (2020) for risk preference heterogeneity in public good funding lotteries.

define

$$F(\alpha, n) := \frac{\partial X^e}{\partial \alpha} = \frac{(n-1)}{c}\left\{\frac{n\alpha V e^{\alpha V} - [e^{2\alpha V} + (n-2)e^{\alpha V} - (n-1)]}{\alpha^2(n + e^{\alpha V} - 1)^2}\right\}. \tag{24}$$

We also define $A := e^{\alpha V}$, where $\alpha \neq 0$, so that $A \in (0,1)\cup(1,\infty)$. We find that $F(\alpha, n) = 0$ if and only if

$$G(A, n) := nA\ln A - [A^2 + (n-2)A - (n-1)] = 0 \tag{25}$$

Suppose that $F(\alpha, n) = 0$ at $\alpha = \alpha^*$ so that $G(A, n) = 0$ at $A = A^* := e^{\alpha^* V}$. Therefore, (25) is equivalent to

$$n\ln A^* = A^* + (n-2) - \frac{(n-1)}{A^*}. \tag{26}$$

Case (i) $n = 2$. Then, (26) can be written as

$$\ln A^* = A^* - \frac{1}{A^*} \tag{27}$$

which has no solution since $A^* \in (0,1)\cup(1,\infty)$. However, we can write $G(A, 2)$ defined in (25) as $2A\ln A - A^2 + 1$ which is strictly positive (negative) whenever $A < (>)\ 1$. Thus, X^e increases as $A \to 1$ (i.e. $\alpha \to 0$). Since X^e in (15) is continuous in α, then X^e is maximized when $\alpha = 0$.

Case (ii) $n > 2$. Then, (26) implies $A^* > n-1$ (i.e., $\alpha^* > \frac{\ln(n-1)}{V} > 0$). Given $F(\alpha, n)$ in (24), we find that

$$\left.\frac{\partial F}{\partial \alpha}\right|_{\alpha=\alpha^*} = \frac{(n-1)Ve^{\alpha V}}{c\alpha^3(n + e^{\alpha V} - 1)^3}\cdot\left.\frac{\partial G}{\partial A}\right|_{A=A^*} \tag{28}$$

where $G(\alpha, n)$ is defined in (25). Thus, $\frac{\partial F}{\partial \alpha}|_{\alpha=\alpha^*} < 0$ if

$$\left.\frac{\partial G}{\partial A}\right|_{A=A^*} = n(1 + \ln A^*) - (2A^* + n - 2) < 0. \tag{29}$$

which is equivalent to

$$n\ln A^* < 2(A^* - 1). \tag{30}$$

We find that

$$2(A-1) > A + (n-2) - \frac{(n-1)}{A} \tag{31}$$

whenever $A > n-1$. Given $A^* > n-1$, then (29) holds and $\frac{\partial F}{\partial \alpha} < 0$ whenever $F(\alpha, n) = 0$. Thus, there exists a unique value of α that maximizes X^e. Given $F(\alpha, n)$ in (24), we find that

$$\left.\frac{\partial F}{\partial n}\right|_{\alpha=\alpha^*} = \frac{(n-1)}{c\alpha^2(n + e^{\alpha V} - 1)^2}\cdot\left.\frac{\partial G}{\partial n}\right|_{A=A^*} \tag{32}$$

and

$$\left.\frac{\partial G}{\partial n}\right|_{A=A^*} = A^*\ln A^* - A^* + 1 > 0. \tag{33}$$

Since $\frac{\partial F}{\partial \alpha}|_{\alpha=\alpha^*} < 0$ and $\frac{\partial F}{\partial n}|_{\alpha=\alpha^*} > 0$, the implicit function theorem suggests that α^* is increasing in n. ■

Appendix B. Proof of Proposition 3

Suppose that the largest value of α_{ik} in group i is α for all $i = 1, \dots, n$ and the second largest value of α_{ik} in group i is β_i. We let $s_i^\alpha(X)$ and $s_i^\beta(X)$ denote the share functions of player ik such that $\alpha_{ik} = \alpha$ and $\alpha_{ik} = \beta_i$, respectively. Thus, for $\tau = \alpha, \beta_i$, $s_i^\tau(X) > 0$ for all $X \in (0, T_\tau)$, where $T_\tau = \frac{V}{c}$ given $\tau = 0$ and $T_\tau = \frac{e^{\tau V}-1}{\tau c}$ given $\tau \neq 0$. Given $\alpha > \beta_i$ and the properties of $s_i(X)$ derived in Lemma 1, we find that there exists $\psi < T_{\beta_i}$ such that $s_i^\alpha(\psi) = s_i^\beta(\psi)$ and $s_i^\alpha(X) > s_i^\beta(X)$ for all $X \in (\psi, T_\alpha)$. We show in each of the following cases that $X^e > \psi$ and, therefore, the representative players of group i in equilibrium are those with $\alpha_{ik} = \alpha$. Since $\max\{\alpha_{i1}, \dots, \alpha_{im}\} = \alpha$ for all $i = 1, \dots, n$, we derive a symmetric equilibrium by setting $s_i^\alpha(X^e) = \frac{1}{n}$ and find that each group invests $\frac{X^e}{n}$ in equilibrium.

Part (i): $0 = \alpha > \beta_i$. According to (11) and (12), $s_i^\alpha(\psi) = s_i^\beta(\psi)$ is equivalent to

$$1 - \frac{\psi}{T_\alpha} = \frac{T_{\beta_i} - \psi}{T_{\beta_i} + (e^{\beta_i V} - 1)\psi} \tag{34}$$

which implies

$$\psi = \frac{\beta_i V e^{\beta_i V} - (e^{\beta_i V} - 1)}{(e^{\beta_i V} - 1)\beta_i c}. \tag{35}$$

It follows that

$$s_i^\alpha(\psi) = s_i^\beta(\psi) = \frac{e^{\beta_i V} - 1 - e^{\beta_i V}}{(e^{\beta_i V} - 1)\beta_i V} > \frac{1}{2}. \tag{36}$$

whenever

$$2e^{\beta_i V} - \beta_i V - \beta_i V e^{\beta_i V} > 2. \tag{37}$$

Define $f(a) = 2e^a - a - ae^a$. We find that $f(0) = 2$ and $f'(a)$ is negative for all $a < 0$. By letting $a = \beta_i V$, we find that $f(\beta_i V) > 2$ and (36) holds. It follows that, $s_i(X) = s_i^\alpha(X)$ whenever $s_i(X) \leq \frac{1}{2}$. Given $n \geq 2$, we find that $s_i(X^e) = \frac{1}{n} \leq \frac{1}{2}$ and $X^e > \psi$. By setting

$$1 - \frac{X^e}{T_\alpha} = \frac{1}{n}, \tag{38}$$

we obtain

$$X^e = \frac{(n-1)V}{nc}. \tag{39}$$

Part (ii): $0 > \alpha > \beta_i$. According to (12), $s_i^\alpha(\psi) = s_i^\beta(\psi)$ is equivalent to

$$\frac{T_\alpha - \psi}{T_\alpha + (e^{\alpha V} - 1)\psi} = \frac{T_{\beta_i} - \psi}{T_{\beta_i} + (e^{\beta_i V} - 1)\psi} \tag{40}$$

which implies

$$\psi = \frac{e^{\alpha V}T_{\beta_i} - e^{\beta_i V}T_\alpha}{e^{\alpha V} - e^{\beta_i V}}. \tag{41}$$

It follows that

$$s_i^\alpha(\psi) = s_i^\beta(\psi) = \frac{1}{1+\xi}, \tag{42}$$

where $\xi = \frac{e^{\alpha V}T_{\beta_i} - e^{\beta_i V}T_\alpha}{T_\alpha - T_{\beta_i}}$. We find that $\xi \gtreqless 1$ is equivalent to $\alpha\left(\frac{e^{\alpha V}+1}{e^{\alpha V}-1}\right) \gtreqless \beta_i\left(\frac{e^{\beta_i V}+1}{e^{\beta_i V}-1}\right)$. We define function $f(a) := a\left(\frac{e^{aV}+1}{e^{aV}-1}\right)$ and find that $f'(a) \gtreqless 0$ if and only if $a \gtreqless 0$. Thus, if $0 > \alpha > \beta_i$, then $\xi < 1$ and $s_i^\alpha(\psi) = s_i^\beta(\psi) > \frac{1}{2}$. It follows that $s_i(X) = s_i^\alpha(X)$ whenever $s_i(X) \leq \frac{1}{2}$. Given $n \geq 2$, we find that $s_i(X^e) = \frac{1}{n} \leq \frac{1}{2}$ and $X^e > \psi$. By setting

$$\frac{T_\alpha - X^e}{T_\alpha + (e^{\alpha V} - 1)X^e} = \frac{1}{n}, \tag{43}$$

we obtain

$$X^e = \frac{(n-1)(e^{\alpha V} - 1)}{(n + e^{\alpha V} - 1)\alpha c}. \tag{44}$$

Part (iii): $\alpha > 0$. We consider two cases, $\beta_i = 0$ and $\beta_i \neq 0$. Case 1: $\alpha > \beta_i = 0$. According to (11) and (12), $s_i^\alpha(\psi) = s_i^\beta(\psi)$ is equivalent to

$$\frac{T_\alpha - \psi}{T_\alpha + (e^{\alpha V} - 1)\psi} = 1 - \frac{\psi}{T_{\beta_i}} \tag{45}$$

which implies

$$\psi = \frac{\alpha V e^{\alpha V} - (e^{\alpha V} - 1)}{(e^{\alpha V} - 1)\alpha c}. \tag{46}$$

Following the same approach in Part (i), we find that

$$s_i^{\alpha}(\psi) = s_i^{\beta}(\psi) = \frac{e^{\alpha V} - 1 - e^{\alpha V}}{(e^{\alpha V} - 1)\alpha V} < \frac{1}{2}. \tag{47}$$

If n is so small that $\frac{1}{n} > s_i^{\alpha}(\psi)$, then $X^e < \psi$. On the other hand, if n is large enough, then $s_i^{\alpha}(\psi) > \frac{1}{n}$ and $X^e > \psi$. We find that $s_i^{\alpha}(\psi) > \frac{1}{n}$ if and only if

$$n > \frac{\alpha V(e^{\alpha V} - 1)}{(e^{\alpha V} - 1) - \alpha V}. \tag{48}$$

Case 2: $\alpha > \beta_i \neq 0$. Following the same approach in Part (ii), we find that ψ is given by

$$s_i^{\alpha}(\psi) = s_i^{\beta}(\psi) = \frac{T_{\alpha} - T_{\beta_i}}{T_{\beta_i}(e^{\alpha V} - 1) - T_{\alpha}(e^{\beta_i V} - 1)}, \tag{49}$$

which can be larger or smaller than $\frac{1}{2}$, depending on α and β_i. However, if n is large enough, then $s_i^{\alpha}(\psi) > \frac{1}{n}$ and $X^e > \psi$. We find that $s_i^{\alpha}(\psi) > \frac{1}{n}$ if and only if

$$n > \frac{(\alpha - \beta_i)(e^{\alpha V} - 1)(e^{\beta_i V} - 1)}{\beta_i(e^{\alpha V} - 1) - \alpha(e^{\beta_i V} - 1)}. \tag{50}$$

For $s_i^{\alpha}(\psi)$ to be greater than $\frac{1}{n}$ for all $i = 1, \ldots, n$, we need

$$n > \frac{(\alpha - \beta)(e^{\alpha V} - 1)(e^{\beta V} - 1)}{\beta(e^{\alpha V} - 1) - \alpha(e^{\beta V} - 1)} \tag{51}$$

where $\beta := \max\{\beta_1, \ldots, \beta_n\}$. Given $X^e > \psi$ in either case, we use (43) to derive X^e as in (44). ∎

Appendix C. Proof of Lemma 2

Part (i): For each player in group i we have $F_{ik}(w_{ik}, s_i|X) = 0$ for $k = 1, \ldots, m$. In addition to these m equations, we impose

$$F_i(w_{ik}, s_i|X) := \sum_{k=1}^{m} w_{ik} - 1 = 0. \tag{52}$$

Therefore, we have a system of $m+1$ equations for $m+1$ variables of interest, i.e., $w_{i1}, \ldots, w_{im}, s_i$. Differentiating the system of equations with respect to X yields

$$\frac{\partial F_{ik}}{\partial w_{ik}} \cdot \frac{dw_{ik}}{dX} + \frac{\partial F_{ik}}{\partial s_i} \cdot \frac{ds_i}{dX} + \frac{\partial F_{ik}}{\partial X} = 0 \tag{53}$$

for $k = 1, \ldots, m$ and

$$\frac{\partial F_i}{\partial w_{ik}} \cdot \frac{dw_{ik}}{dX} + \frac{\partial F_i}{\partial s_i} \cdot \frac{ds_i}{dX} + \frac{\partial F_i}{\partial X} = 0. \tag{54}$$

In matrix form, we have

$$\begin{bmatrix} \frac{\partial F_{i1}}{\partial w_{i1}} & 0 & 0 & \ldots & 0 & \frac{\partial F_{i1}}{\partial s_i} \\ 0 & \frac{\partial F_{i2}}{\partial w_{i2}} & 0 & \ldots & 0 & \frac{\partial F_{i2}}{\partial s_i} \\ 0 & 0 & \frac{\partial F_{i3}}{\partial w_{i3}} & \ldots & 0 & \frac{\partial F_{i3}}{\partial s_i} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{\partial F_{im}}{\partial w_{im}} & \frac{\partial F_{im}}{\partial s_i} \\ 1 & 1 & 1 & \ldots & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{dw_{i1}}{dX} \\ \frac{dw_{i2}}{dX} \\ \frac{dw_{i3}}{dX} \\ \vdots \\ \frac{dw_{im}}{dX} \\ \frac{ds_i}{dX} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F_{i1}}{\partial X} \\ -\frac{\partial F_{i2}}{\partial X} \\ -\frac{\partial F_{i3}}{\partial X} \\ \vdots \\ -\frac{\partial F_{im}}{\partial X} \\ 0 \end{bmatrix} \tag{55}$$

and using Cramer's rule, we find

$$\frac{ds_i}{dX} = -\left[\sum_{k=1}^{m}\left(\frac{\partial F_{ik}}{\partial X}\prod_{l\neq k}\frac{\partial F_{il}}{\partial w_{il}}\right) \Big/ \sum_{k=1}^{m}\left(\frac{\partial F_{ik}}{\partial s_i}\prod_{l\neq k}\frac{\partial F_{il}}{\partial w_{il}}\right)\right]. \tag{56}$$

Given the definition in (20), we find that

$$\frac{\partial F_{ik}}{\partial X} = -g''(w_{ik}s_iX)w_{ik}s_iX - g'(w_{ik}s_iX) < 0 \tag{57}$$

$$\frac{\partial F_{ik}}{\partial w_{ik}} = -g''(w_{ik}s_iX)s_iX^2 < 0 \tag{58}$$

$$\frac{\partial F_{ik}}{\partial s_i} = \begin{cases} -V - g''(w_{ik}s_iX)w_{ik}X^2 < 0 & \text{if } \alpha_{ik} = 0 \\ -\frac{e^{\alpha_{ik}V}(e^{\alpha_{ik}V}-1)}{(1-s_i+s_ie^{\alpha_{ik}V})^2\alpha_{ik}} - g''(w_{ik}s_iX)w_{ik}X^2 < 0 & \text{if } \alpha_{ik} \neq 0 \end{cases} \tag{59}$$

for $k = 1, \ldots, m$. It follows that the ratio inside the brackets in (56) is positive for any m (regardless of whether m is odd or even). Therefore, $\frac{ds_i}{dX} < 0$ and group i's share function is strictly decreasing in X.

Part (ii): Given $X = 0$ and $g'(0) = 0$, we find from (20) that $F_{ik}(w_{ik}, s_i|X) = 0$ when $s_i = 1$.

Part (iii): The first-order condition in (19) is equivalent to

$$g'(w_{ik}s_iX) = \begin{cases} \frac{(1-s_i)V}{X} & \text{if } \alpha_{ik} = 0 \\ \frac{(1-s_i)(e^{\alpha_{ik}V}-1)}{(1-s_i+s_ie^{\alpha_{ik}V})\alpha_{ik}X} & \text{if } \alpha_{ik} \neq 0 \end{cases} \tag{60}$$

Given $s_i < 1$, the right-hand side of (60) converges to zero as $X \to \infty$. Since $g'' > 0$ and $g'(0) = 0$, the left-hand side of (60) converges to zero as $s_i \to 0$. ∎

Appendix D. Proof of Proposition 6

Using the relative deviations to redefine player ik's investment and risk level, $x_{ik}^* = \bar{x}(1 - b_{ik})$ and $\alpha_{ik} = \bar{\alpha}(1 + a_{ik})$ respectively, first-order condition (18) holds with equality in equilibrium and can be written as

$$\frac{\sum_{i,k}(1-b_{ik}) - \sum_k(1-b_{ik})}{\left[\sum_{i,k}(1-b_{ik})\right]^2}\bar{\beta}_{ik}$$
$$= \bar{x}\bar{\alpha}(1+a_{ik})g'(\bar{x}(1-b_{ik}))\left[\frac{\sum_k(1-b_{ik})}{\sum_{i,k}(1-b_{ik})}\bar{\beta}_{ik} + 1\right], \tag{61}$$

where $\bar{\beta}_{ik} := e^{\bar{\alpha}V(1+a_{ik})} - 1$. Multiplying both sides by $[\sum_{i,k}(1-b_{ik})]^2$, using the definitions of k_2 and k_3 in Proposition 6, and Taylor-expanding both sides of (61) to the second order in powers of ρ, obtain the following expressions for the left-hand side (LHS) and right-hand side (RHS)

$$\begin{aligned} LHS =& [m(n-1) - B + B_i] \\ &\times \left[\bar{\beta} + \bar{\alpha}V(\bar{\beta}+1)a_{ik} + \frac{1}{2}\bar{\alpha}^2V^2(\bar{\beta}+1)a_{ik}^2\right] + O(S_a^3) \\ =& m(n-1)\bar{\beta} + \bar{\alpha}Vm(n-1)(\bar{\beta}+1)a_{ik} \\ &+ \frac{1}{2}\bar{\alpha}^2V^2m(n-1)(\bar{\beta}+1)a_{ik}^2 \\ &+ \bar{\beta}B - \bar{\alpha}V(\bar{\beta}+1)Ba_{ik} + \bar{\beta}B_i + \bar{\alpha}V(\bar{\beta}+1)B_ia_{ik} + O(S_a^3). \\ RHS =& \bar{\alpha}\bar{x}g'(1+a_{ik})(1 - k_2b_{ik} + \frac{1}{2}k_2k_3b_{ik}^2)(nm-B) \\ &\times \left\{(m-B_i)\left[\bar{\beta} + \bar{\alpha}V(\bar{\beta}+1)a_{ik} + \frac{1}{2}\bar{\alpha}^2V^2(\bar{\beta}+1)a_{ik}^2\right]\right. \\ &\left. + (nm-B)\right\} + O(S_a^3). \end{aligned} \tag{62}$$

$$
\begin{aligned}
&= \bar{\alpha}\bar{x}g'nm^2(\bar{\beta}+n) + \bar{\alpha}^2\bar{x}g'nm^2V(\bar{\beta}+1)a_{ik} \\
&+ \frac{1}{2}\bar{\alpha}^3\bar{x}g'nm^2V^2(\bar{\beta}+1)a_{ik}^2 - \bar{\alpha}\bar{x}g'nm\bar{\beta}B_i \\
&- \bar{\alpha}^2\bar{x}g'nmV(\bar{\beta}+1)B_i a_{ik} - \bar{\alpha}\bar{x}g'm(2n+\bar{\beta})B \\
&- \bar{\alpha}^2\bar{x}g'mV(\bar{\beta}+1)Ba_{ik} + \bar{\alpha}\bar{x}g'\bar{\beta}BB_i \\
&+ \bar{\alpha}\bar{x}g'B^2 - k_2\bar{\alpha}\bar{x}g'nm^2(\bar{\beta}+n)b_{ik} \\
&- k_2\bar{\alpha}^2\bar{x}g'nm^2V(\bar{\beta}+1)a_{ik}b_{ik} + k_2\bar{\alpha}\bar{x}g'nm\bar{\beta}B_i b_{ik} \\
&+ k_2\bar{\alpha}\bar{x}g'm(2n+\bar{\beta})Bb_{ik} + \frac{1}{2}k_2k_3\bar{\alpha}\bar{x}g'nm^2(\bar{\beta}+n)b_{ik}^2 \\
&+ \bar{\alpha}\bar{x}g'nm^2(\bar{\beta}+n)a_{ik} \\
&+ \bar{\alpha}^2\bar{x}g'nm^2V(\bar{\beta}+1)a_{ik}^2 - \bar{\alpha}\bar{x}g'nm\bar{\beta}B_i a_{ik} \\
&- \bar{\alpha}\bar{x}g'm(2n+\bar{\beta})Ba_{ik} \\
&- k_2\bar{\alpha}\bar{x}g'nm^2(\bar{\beta}+n)a_{ik}b_{ik} + O(S_a^3).
\end{aligned}
$$

Note, g', g'' and g''' are shorthand for those functions evaluated at $\bar{x}$. Also, recall that we seek to find an approximation for individual equilibrium investment of the form $x_{ik} = x_{ik}^{(1)} + x_{ik}^{(2)} + O(S_a^3)$, where $x_{ik}^{(1)}$ and $x_{ik}^{(2)}$ represent first and second-order corrections, respectively, and $O(S_a^2)$ the approximation error. Writing equilibrium investments of player ik as small deviations $b_{ik} \ll 1$ from the symmetric level $\bar{x}$, implies that it is sufficient to find the approximate investment level deviation $b_{ik} = b_{ik}^{(1)} + b_{ik}^{(2)} + O(S_a^3)$. We denote the associated s-order ($s = 1, 2$) aggregate corrections as $B^{(s)}$ and group-level aggregate corrections as $B_i^{(s)}$. Equating $O(\rho^0)$ terms on both sides of (62), obtain the zero-order approximation, i.e., the first-order condition equality (21) for the symmetric investment level $\bar{x}$, which can be written

$$
m(n-1)\bar{\beta} = \bar{\alpha}\bar{x}g'nm^2(\bar{\beta}+n). \tag{63}
$$

For the first-order approximation, equate $O(\rho^1)$ terms on both sides of (62). Collecting like terms, obtain

$$
\begin{aligned}
&\frac{m(n-1)}{\bar{\beta}+n}\left[\bar{\alpha}nV(\bar{\beta}+1) - \bar{\beta}(\bar{\beta}+n)\right]a_{ik} \\
&= \left[\bar{\beta} - \bar{\alpha}\bar{x}g'm(2n+\bar{\beta})\right]B^{(1)} \\
&- \left[\bar{\beta} + \bar{\alpha}\bar{x}g'm\bar{\beta}\right]B_i^{(1)} - k_2m(n-1)\bar{\beta}b_{ik}^{(1)}.
\end{aligned} \tag{64}
$$

Summing both sides over players $k = 1, \ldots, m$ in group i, then summing over all groups $i = 1, \ldots, n$, using equality (63) and simplifying, obtain

$$
\left[\bar{\alpha}nV(\bar{\beta}+1) - \bar{\beta}(\bar{\beta}+n)\right]A = -\bar{\beta}(1+k_2)B^{(1)}. \tag{65}
$$

By construction, $A = \sum_{i,k} a_{ik} = 0$, and hence, the first-order correction to aggregate investment, $B^{(1)}$, is equal to zero. Therefore, the second-order correction is needed.

Summing (64) over all players $k = 1, \ldots, m$ in group i, using equality (63), and using the fact that $B^{(1)} = 0$, obtain

$$
\begin{aligned}
&\frac{m(n-1)}{\bar{\beta}+n}\left[\bar{\alpha}nV(\bar{\beta}+1) - \bar{\beta}(\bar{\beta}+n)\right]A_i \\
&= -\left[m\bar{\beta} + \frac{m(n-1)\bar{\beta}^2}{(\bar{\beta}+n)}\right]B_i^{(1)} - k_2m(n-1)\bar{\beta}B_i^{(1)}.
\end{aligned} \tag{66}
$$

Solving (66) for the first-order correction of aggregate investment of group i, obtain

$$
B_i^{(1)} = -\frac{\delta(n-1)}{\bar{\beta}\gamma}A_i, \tag{67}
$$

where $\gamma := n(\bar{\beta}+1) + k_2(n-1) + n$ and $\delta := n(\bar{\beta}+1)\ln(\bar{\beta}+1) - \bar{\beta}(\bar{\beta}+n)$.[21] Plugging this into (64) and simplifying, obtain first-order correction for player ik

$$
b_{ik}^{(1)} = \frac{\delta}{k_2\bar{\beta}(\bar{\beta}+n)}\left[\frac{n(\bar{\beta}+1)}{m\gamma}A_i - a_{ik}\right]. \tag{68}
$$

Therefore, in the linear approximation, player ik's correction to the symmetric investment level $\bar{x}$ is a linear combination of her own risk level and the aggregate risk level of her group. Interestingly, when $\frac{\delta}{k_2\bar{\beta}(\bar{\beta}+n)} < 0$, $b_{ik}^{(1)}$ is decreasing in A_i and increasing in a_{ik}. Put differently, in this case, player ik free-rides more and more as the aggregate level of risk of her group members increases.

Now, we obtain the second-order correction to aggregate investment $B^{(2)}$. Equating $O(\rho^2)$ terms from both sides of (62), using the fact that $B^{(1)} = 0$, and collecting like terms, obtain

$$
\begin{aligned}
&\frac{m(n-1)(\bar{\beta}+1)\ln(\bar{\beta}+1)}{2(\bar{\beta}+n)}\left[n\ln(\bar{\beta}+1) - 2\bar{\beta}\right]a_{ik}^2 \\
&= \left[\bar{\beta} - \bar{\alpha}\bar{x}g'm(2n+\bar{\beta})\right]B^{(2)} \\
&- \frac{1}{\bar{\beta}+n}\left[n(\bar{\beta}+1)^2\ln(\bar{\beta}+1) + \bar{\beta}^2(n-1)\right]B_i^{(1)}a_{ik} \\
&- \frac{k_2m\bar{\beta}(n-1)}{\bar{\beta}+n}\left[(\bar{\beta}+1)\ln(\bar{\beta}+1) + (\bar{\beta}+n)\right]b_{ik}^{(1)}a_{ik} \\
&- k_2m\bar{\beta}(n-1)b_{ik}^{(2)} + \frac{k_2\bar{\beta}^2(n-1)}{\bar{\beta}+n}B_i^{(1)}b_{ik}^{(1)} \\
&+ \frac{1}{2}k_2k_3m\bar{\beta}(n-1)(b_{ik}^{(1)})^2 - \left[\bar{\beta} + \bar{\alpha}\bar{x}g'nm\bar{\beta}\right]B_i^{(2)}.
\end{aligned} \tag{69}
$$

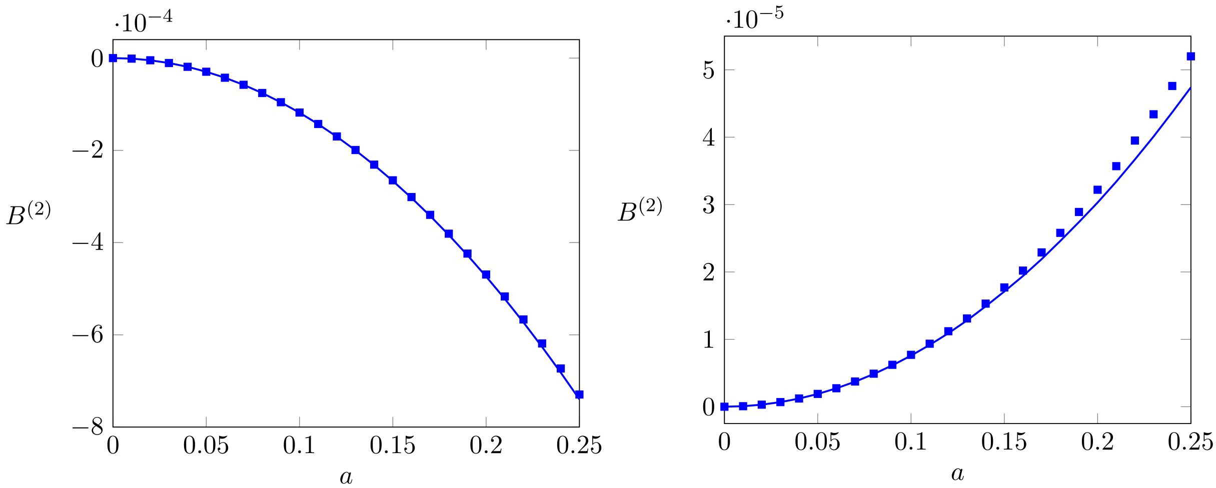
Summing (69) over players $k = 1, \ldots, m$ in group i, then summing over all groups $i = 1, \ldots, n$, simplifying and collecting like terms, obtain

$$
\begin{aligned}
&\frac{m(n-1)(\bar{\beta}+1)\ln(\bar{\beta}+1)\left[n\ln(\bar{\beta}+1) - 2\bar{\beta}\right]}{2(\bar{\beta}+n)}\sum_{i,k}a_{ik}^2 \\
&= -m\bar{\beta}(n-1)(1+k_2)B^{(2)} \\
&- \frac{\left[n(\bar{\beta}+1)^2\ln(\bar{\beta}+1) + \bar{\beta}^2(n-1)\right]}{\bar{\beta}+n}\sum_i B_i^{(1)}A_i \\
&- \frac{k_2m\bar{\beta}(n-1)\left[(\bar{\beta}+1)\ln(\bar{\beta}+1) + (\bar{\beta}+n)\right]}{\bar{\beta}+n}\sum_{i,k}b_{ik}^{(1)}a_{ik} \\
&+ \frac{k_2\bar{\beta}^2(n-1)}{\bar{\beta}+n}\sum_i(B_i^{(1)})^2 + \frac{k_2k_3m\bar{\beta}(n-1)}{2}\sum_{i,k}(b_{ik}^{(1)})^2.
\end{aligned} \tag{70}
$$

Note that each summation in (70) can be written as linear combinations of summation terms $\sum_{i,k} a_{ik}^2$ and $\sum_i A_i^2$ as follows

$$
\begin{aligned}
&\sum_i B_i^{(1)}A_i = -\frac{\delta(n-1)}{\gamma\bar{\beta}}\sum_i A_i^2. \\
&\sum_{i,k} b_{ik}^{(1)}a_{ik} = \frac{n\delta(\bar{\beta}+1)}{k_2m\gamma\bar{\beta}(\bar{\beta}+n)}\sum_i A_i^2 - \frac{\delta}{k_2\bar{\beta}(\bar{\beta}+n)}\sum_{i,k}a_{ik}^2. \\
&\sum_i (B_i^{(1)})^2 = \left[\frac{\delta(n-1)}{\gamma\bar{\beta}}\right]^2\sum_i A_i^2. \\
&\sum_i (b_{ik}^{(1)})^2 = \frac{n(\bar{\beta}+1)\left[n(\bar{\beta}+1) - 2\gamma\right]}{m\gamma^2}\left[\frac{\delta}{k_2\bar{\beta}(\bar{\beta}+n)}\right]^2\sum_i A_i^2 \\
&\qquad + \left[\frac{\delta}{k_2\bar{\beta}(\bar{\beta}+n)}\right]^2\sum_{i,k}a_{ik}^2.
\end{aligned} \tag{71}
$$

[21] Recall that $\bar{\beta} = e^{\bar{\alpha}V} - 1$, implying $\bar{\alpha}V = \ln(\bar{\beta}+1)$.



(a) $g(x_i) = x_i^2$ (b) $g(x_i) = \frac{x_i}{10} \int_{1-x_i}^{1} t^{-1} \exp(t)dt$

Fig. 10. Aggregate relative investment deviation, $B^{(2)}$, from the symmetric level as a function of the degree of heterogeneity, a, in the contest between $n = 4$ individuals. Individual i has risk level $\alpha_i = \bar{\alpha}(1 + a_i)$, $\bar{\alpha} = -1.5$, and heterogeneity parameters a_i are as follows: $a_1 = \frac{3}{4}a$, $a_2 = \frac{1}{4}a$, $a_3 = -\frac{1}{4}a$, and $a_4 = -\frac{3}{4}a$. The quadratic approximation is given by the solid curves, and solid squares indicate numerical solutions.

$$\lambda_a = \frac{mn\left\{\delta\left[k_3\delta + 2k_2\bar{\beta}(\bar{\beta}+n)\right] + k_2\bar{\beta}(\bar{\beta}+1)\ln(\bar{\beta}+1)\left[2(\delta+\bar{\beta}(\bar{\beta}+n)) - n(\bar{\beta}+n)\ln(1+\bar{\beta})\right]\right\}}{2k_2\bar{\beta}^2(1+k_2)(\bar{\beta}+n)^2},$$

$$\lambda_A = \frac{n\delta}{2k_2m\bar{\beta}^2\gamma^2(1+k_2)(\bar{\beta}+n)^2}\left\{k_3n\delta(n-2\gamma) - 2k_2\gamma\bar{\beta}^3 + 2n\bar{\beta}\left[k_2^2\delta(n-1)^2 - k_2n\gamma + k_3\delta(n-\gamma)\right]\right.$$
$$\left. + \bar{\beta}^2\left[\delta(2k_2^2(n-1)^2 + k_3n^2) - 4k_2n\gamma\right] + 2k_2\gamma n^2(\bar{\beta}+1)^2\ln(\bar{\beta}+1)\right\}$$

Box III.

Plugging summations (71) into (70), recalling notation for sample variance of risk level across groups and across individuals, $S_A^2 = \frac{1}{n}\sum_i A_i$ and $S_a^2 = \frac{1}{nm}\sum_{i,k} a_{ik}$ respectively, solving for the second-order correction to aggregate investment $B^{(2)}$ where λ_a and λ_A can be simplified as in Box III. ■

Appendix E. Heterogeneity and aggregate investment when *m* = 1

When $m = 1$, the group contest reduces to a contest between n individuals, and the aggregate deviation from the symmetric equilibrium level in Proposition 6 can be rewritten as $B^{(2)} = \lambda S_a^2$, where λ is equal to the sum of coefficients λ_a and λ_A, each evaluated at $m = 1$. Thus, when $m = 1$, $S_a^2 = S_A^2$, sorting is no longer relevant, and the deviation from the symmetric aggregate equilibrium investment level is linear in the sample variance of risk level, S_a^2, without an intercept. Increasing the variance in risk level across individuals can lead to an increase or decrease in aggregate investment: when $\lambda < 0$, increasing S_a^2 decreases $B^{(2)}$, leading to an increase in X^e; vice versa when $\lambda > 0$. Coefficient λ is a complicated function of model parameters, but we are able to show that the steepness of cost function $g(x)$ plays a critical role in determining the effect of heterogeneity.

Fig. 10 plots $B^{(2)}$ as a function of the degree of heterogeneity, a, in the contest between $n = 4$ individuals. In each panel, the average risk level is $\bar{\alpha} = -1.5$, individual risk levels are given by $\alpha_i = \bar{\alpha}(1 + a_i)$, and individual heterogeneity parameters are as follows: $a_1 = \frac{3}{4}a$, $a_2 = \frac{1}{4}a$, $a_3 = -\frac{1}{4}a$, and $a_4 = -\frac{3}{4}a$. The quadratic approximation is given by the solid curves, and solid squares indicate numerical solutions. In panel (a) we consider a standard quadratic investment cost, $g(x_i) = x_i^2$, while in panel (b) we consider a much more "explosive" cost of effort, $g(x_i) = x_i \int_{1-x_i}^{1} t^{-1}\exp(t)dt/10$. Under relatively moderate investment costs (panel (a)), $\lambda < 0$, and there is a positive effect of heterogeneity on X^e.[22] In contrast, when costs grow rapidly, $\lambda > 0$, and there is a negative effect of heterogeneity on X^e. The direction of the effect is determined by the individual second-order correction $b_{ik}^{(2)}$. Players with identical $|\alpha_{ik}|$ distributed symmetrically about $\bar{\alpha}$ have opposing and potentially different effects on X^e. For the example depicted in Fig. 10, heterogeneity leads players with $\alpha_i > \bar{\alpha}$ to increase their investment level, while players with $\alpha_i < \bar{\alpha}$ decrease investment. The sum of these opposing effects is positive and increasing in heterogeneity in panel (a), and vice versa in panel (b).

References

Abbink, Klaus, Brandts, Jordi, Herrmann, Benedikt, Orzen, Henrik, 2010. Intergroup conflict and intra-group punishment in an experimental contest game. Amer. Econ. Rev. 420–447.
Akerlof, Robert J., Holden, Richard T., 2012. The nature of tournaments. Econom. Theory 51 (2), 289–313.
Albert, Steven M., Duffy, John, 2012. Differences in risk aversion between young and older adults. Neurosci. Neuroecon. 2012 (1).
Baik, Kyung Hwan, 1993. Effort levels in contests: The public-good prize case. Econom. Lett. 41 (4), 363–367.
Baik, Kyung Hwan, 2008. Contests with group-specific public-good prizes. Soc. Choice Welf. 30 (1), 103–117.

[22] This result holds for any power cost function $g(x) = cx^a$, with $a > 1$.

Baik, Kyung Hwan, Kim, In-Gyu, Na, Sunghyun, 2001. Bidding for a group-specific public-good prize. J. Publ. Econ. 82 (3), 415–429.
Baik, Kyung Hwan, Shogren, Jason F., 1998. A behavioral basis for best-shot public-good contests. Adv. Appl. Microecon.: Contests 7, 169–178.
Barbieri, Stefano, Malueg, David A., Topolyan, Iryna, 2014. The best-shot all-pay (group) auction with complete information. Econom. Theory 57 (3) 603–640.
Baye, Michael R., Kovenock, Dan, De Vries, Casper G., 1996. The all-pay auction with complete information. Econom. Theory 8 (2), 291–305.
Borghans, Lex, Heckman, James J, Golsteyn, Bart HH, Meijers, Huub, 2009. Gender differences in risk aversion and ambiguity aversion. J. Eur. Econom. Assoc. 7 (2–3), 649–658.
Bos, Olivier, 2011. How lotteries outperform auctions. Econom. Lett. 110 (3), 262–264.
Brookins, Philip, Lightle, John P., Ryvkin, Dmitry, 2015a. An experimental study of sorting in group contests. Labour Econ. 35, 16–25.
Brookins, Philip, Lightle, John P., Ryvkin, Dmitry, 2015b. Optimal sorting in group contests with complementarities. J. Econ. Behav. Organ. 112, 311–323.
Brookins, Philip, Lightle, John P., Ryvkin, Dmitry, 2018. Sorting and communication in weak-link group contests. J. Econ. Behav. Organ. 152, 64–80.
Charness, Gary, Gneezy, Uri, 2012. Strong evidence for gender differences in risk taking. J. Econ. Behav. Organ. 83 (1), 50–58.
Chen, Hua, Lim, Noah, 2013. Should managers use team-based contests? Manage. Sci. 59 (12), 2823–2836.
Chen, Zhuoqiong Charlie, Ong, David, Segev, Ella, 2017. Heterogeneous risk/loss aversion in complete information all-pay auctions. Eur. Econ. Rev. 95, 23–37.
Chowdhury, Subhasish M., Lee, Dongryul, Sheremeta, Roman M., 2013. Top guns may not fire: Best-shot group contests with group-specific public good prizes. J. Econ. Behav. Organ. 92, 94–103.
Chowdhury, Subhasish M., Lee, Dongryul, Topolyan, Iryna, 2016. The max-min group contest: Weakest-link (group) all-pay auction. South. Econ. J. 83 (1), 105–125.
Chowdhury, Subhasish M., Topolyan, Iryna, 2016. The attack-and-defense group contests: Best shot versus weakest link. Econ. Inq. 54 (1), 548–557.
Congleton, Roger D., Hillman, Arye L., Konrad, Kai A., 2008. 40 Years of Research on Rent Seeking 2: Applications: Rent Seeking in Practice, Vol. 2. Springer Science & Business Media.
Connelly, Brian L, Tihanyi, Laszlo, Crook, T Russell, Gangloff, K Ashley, 2014. Tournament theory: Thirty years of contests and competitions. J. Manage. 40 (1), 16–47.
Corchón, Luis C., Serena, Marco, 2018. Contest theory. In: Handbook of Game Theory and Industrial Organization, Vol. II. Edward Elgar Publishing.
Cornes, Richard, Hartley, Roger, 2003. Risk aversion, heterogeneity and contests. Publ. Choice 117 (1–2), 1–25.
Cornes, Richard, Hartley, Roger, 2005. Asymmetric contests with general technologies. Econ. Theory 26 (4), 923–946.
Cornes, Richard, Hartley, Roger, 2012. Risk aversion in symmetric and asymmetric contests. Econom. Theory 51 (2), 247–275.
Croson, Rachel, Gneezy, Uri, 2009. Gender differences in preferences. J. Econ. Literature 47 (2), 448–474.
Dechenaux, Emmanuel, Kovenock, Dan, Sheremeta, Roman M., 2015. A survey of experimental research on contests, all-pay auctions and tournaments. Exp. Econ. 18 (4), 609–669.
Deck, Cary, Schlesinger, Harris, 2014. Consistency of higher order risk preferences. Econometrica 82 (5), 1913–1943.
Ebert, Sebastian, Wiesen, Daniel, 2014. Joint measurement of risk aversion, prudence, and temperance. J. Risk Uncertainty 48, 231–252.
Eeckhoudt, Louis, Schlesinger, Harris, 2006. Putting risk in its proper place. Amer. Econ. Rev. 96 (1), 280–289.
Esteban, Joan, Ray, Debraj, 2001. Collective action and the group size paradox. Am. Polit. Sci. Rev. 95 (3), 663–672.
Fibich, Gadi, Gavious, Arieh, Sela, Aner, 2006. All-pay auctions with risk-averse players. Internat. J. Game Theory 34 (4), 583–599.
Green, Jerry R., Stokey, Nancy L., 1983. A comparison of tournaments and contracts. J. Polit. Econ. 91 (3), 349–364.
Hillman, Arye L., Samet, Dov, 1987. Dissipation of contestable rents by small numbers of contenders. Publ. Choice 54 (1), 63–82.
Jindapon, Paan, Whaley, Christopher A., 2015. Risk lovers and the rent over-investment puzzle. Publ. Choice 164 (1–2), 87–101.
Jindapon, Paan, Yang, Zhe, 2017. Risk attitudes and heterogeneity in simultaneous and sequential contests. J. Econ. Behav. Organ. 138, 69–84.
Jindapon, Paan, Yang, Zhe, 2020. Free riders and the optimal prize in public-good funding lotteries. J. Publ. Econ. Theory 22 (5), 1289–1312.
Kalra, Ajay, Shi, Mengze, 2001. Designing optimal sales contests: A theoretical perspective. Mark. Sci. 20 (2), 170–193.
Katz, Eliakim, Nitzan, Shmuel, Rosenberg, Jacob, 1990. Rent-seeking for pure public goods. Publ. Choice 65 (1), 49–60.
Kimball, Miles, 1990. Precautionary saving in the small and in the large. Econometrica 58, 53–73.
Kolmar, Martin, Rommeswinkel, Hendrik, 2013. Contests with group-specific public goods and complementarities in efforts. J. Econ. Behav. Organ. 89, 9–22.
Konrad, Kai A., 2009. Strategy and dynamics in contests. Oxford University Press.
Konrad, Kai A., Schlesinger, Harris, 1997. Risk aversion in rent-seeking and rent-augmenting games. Econom. J. 107, 1671–1683.
Krishna, Vijay, Morgan, John, 1998. The winner-take-all principle in small tournaments. Adv. Appl. Microecon. 7, 61–74.
Lawler, Edward E., Mohrman, Susan Albers, Benson, George, 2001. Organizing for High Performance: Employee Involvement, TQM, Reengineering, and Knowledge Management in the Fortune 1000: The CEO Report. Jossey-Bass.
Lawler, Edward E., Mohrman, Susan Albers, Ledford, Gerald E., 1995. Creating High Performance Organizations: Practices and Results of Employee Involvement and Total Quality Management in Fortune 1000 Companies. Jossey-Bass San Francisco, CA.
Lazear, Edward P., Rosen, Sherwin, 1981. Rank-order tournaments as optimum labor contracts. J. Polit. Econ. 89 (5), 841–864.
Lazear, Edward P., Shaw, Kathryn L., 2007. Personnel economics: The economist's view of human resources. J. Econ. Perspect. 21 (4), 91–114.
Lee, Dongryul, 2012. Weakest-link contests with group-specific public good prizes. Eur. J. Political Econ. 28 (2), 238–248.
Li, Jia, Lim, Noah, Chen, Hua, 2019. Examining salesperson effort allocation in teams: A randomized field experiment. Mark. Sci..
March, Christoph, Sahm, Marco, 2018. Contests as selection mechanisms: The impact of risk aversion. J. Econ. Behav. Organ. 150, 114–131.
Menezes, Carmen, Geiss, Charles, Tressler, John, 1980. Increasing downside risk. Amer. Econ. Rev. 70 (5), 921–932.
Modica, Salvatore, Scarsini, Marco, 2005. A note on comparative downside risk aversion. J. Econom. Theory 122 (2), 267–271.
Moldovanu, Benny, Sela, Aner, 2001. The optimal allocation of prizes in contests. Amer. Econ. Rev. 91 (3), 542–558.
Nalebuff, Barry J., Stiglitz, Joseph E., 1983. Prizes and incentives: towards a general theory of compensation and competition. Bell J. Econ. 21–43.
Nitzan, Shmuel, 1991. Collective rent dissipation. Econ. J. 101 (409), 1522–1534.
Nitzan, Shmuel, Ueda, Kaoru, 2014. Intra-group heterogeneity in collective contests. Social Choice Welfare 43 (1), 219–238.
Noussair, Charles N., Trautmann, Stefan T., van de Kuilen, Gijs, 2014. Higher order risk attitudes, demographics, and financial decisions. Rev. Econom. Stud. 81 (1), 325–355.
Nti, Kofi O., 1998. Effort and performance in group contests. Eur. J. Political Econ. 14 (4), 769–781.
O'Keeffe, Mary, Viscusi, W.Kip, Zeckhauser, Richard J., 1984. Economic contests: Comparative reward schemes. J. Labor Econ. 2 (1), 27–56.
Parreiras, Sérgio O., Rubinchik, Anna, 2010. Contests with three or more heterogeneous agents. Games Econom. Behav. 68 (2), 703–715.
Ryvkin, Dmitry, 2011. The optimal sorting of players in contests between groups. Games Econom. Behav. 73 (2), 564–572.
Sahm, Marco, 2017. Risk aversion and prudence in contests. Econ. Bull. 37 (2), 1122–1132.
Schroyen, Fred, Treich, Nicolas, 2016. The power of money: Wealth effects in contests. Games Econom. Behav. 100 (1), 46–68.
Sheremeta, Roman M., 2018. Behavior in group contests: A review of experimental research. J. Econ. Surv. 32 (3), 683–704.
Siegel, Ron, 2009. All-pay contests. Econometrica 77 (1), 71–92.
Skaperdas, Stergios, Gan, Li, 1995. Risk aversion in contests. Econom. J. 105, 951–962.
Topolyan, Iryna, 2014. Rent-seeking for a public good with additive contributions. Soc. Choice Welf. 42 (2), 465–476.
Treich, Nicolas, 2010. Risk-aversion and prudence in rent-seeking games. Publ. Choice 145 (3–4), 339–349.
Trevisan, F., 2020. Optimal prize allocations in group contests. Soc. Choice Welf.
Tullock, Gordon, 1980. Efficient rent seeking. In: Buchanan, James M., Tollison, Robert D., Tullock, Gordon (Eds.), Toward a Theory of the Rent-Seeking Society. Texas A&M University Press, College Station, pp. 97–112.